BLUFFTON UNIVERSITY

MATHEMATICS CONTEST 2019
ANSWER KEY

PART I

Section A (1 point each)

1. 12
2. chord
3. A B (C) D E
4. $\frac{1}{9}$
5. -4
6. 3 and 11 ***(both)***
7. 1019 or 3019 ***(both)***

Section B (2 points each)

8. -673
9. $(5, 1)$
10. $y = -2019$
11. 0%
12. $(-5, 5)$

Section C (3 points each)

13. $c = 6$
14. $38 + 10\sqrt{3}$ units
15. i ***or*** $0 + 1i$ ***or*** $0 + i$

PART II

Section A (1 point each)

1. 1352 or 4040 centimeters ***(both)***
2. 38
3. 11
4. $\frac{39}{41}$
5. MDCLXVI
6. $\pi - \sqrt{2}$ ***or*** $-\sqrt{2} + \pi$
7. 51.6%

Section B (2 points each)

8. $10\sqrt{261}$ ***or*** $30\sqrt{29}$ square units
9. $\theta = 35°$
10. -64 ***or*** -100
11. 17.89 units
12. -90

Section C (3 points each)

13. 22.5 mph
14. $(a^2 - 8a + 11)(a^2 - 8a - 11)$
15. $B - 2A$

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Simplify: $2^2 + 3^1 + 4x^0 + (5x)^0$

Solution: $2^2 + 3^1 + 4x^0 + (5x)^0 = 4 + 3 + 4 \cdot 1 + 1 = 12.$

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2. Give the geometric name of the line segment that connects two points on a circle.

Solution: A segment connecting two points of a circle is called a ***chord***.

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3. On your answer sheet, circle the letters of all sets of numbers which are ***closed under addition***:

A) $\{-1, 0, 1\}$ B) Odd integers C) Even integers
D) Prime numbers E) $\{\ldots, -3, -2, -1, 1, 2, 3, \ldots\}$

Solution: A set is "closed under addition" if, given any two items x and y from the set, $x + y$ is also in the set. (Note that the "two items" do not need to be distinct; that is, we could have $x = y$.) From this list of options, ***only set C*** (the even integers) satisfies this condition. For A, $1 + 1 = 2$ is not in the set. For B, any two odd integers add to give an even integer (therefore not in the set). For D, any two odd primes sum to an even (composite) number. For E, $-1 + 1 = 0$, which is not in the set.

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4. Let a, b, c, d be the numbers $2, 0, 1, 9$ (all four numbers, but not necessarily in that order). What is the ***smallest positive rational number*** that can be expressed by $\dfrac{a-b}{c-d}$?

Solution: To make the smallest possible positive rational number (fraction), we want a large denominator and a small numerator. Considering the possible arrangements of $2, 0, 1, 9$, we find that the smallest possible positive fraction is $\frac{1}{9}$, expressible either as $\frac{2-1}{9-0}$ or $\frac{1-2}{0-9}$.

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5. If $f(x) = x^2 + 2$, and $g(x) = x - 1$, find the value of $f(g(3)) - g(f(3))$.

Solution: $f(g(3)) = f(2) = 6$, and $g(f(3)) = g(11) = 10$, so $f(g(3)) - g(f(3)) = -4$.

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6. The average of 3 and 7 (two distinct prime numbers) is also a prime number—namely, 5. Find two more distinct prime numbers, both less than 17, such that their average is also a prime number.

Solution: The primes less than 17 are $2, 3, 5, 7, 11, 13$. (Note: 1 is ***not*** a prime number.) Clearly, we cannot use 2, because its average with an odd prime would not be an integer. Trying the other combinations, we find the only other answer is $\frac{3+11}{2} = 7$.

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7. Solve for all real values of x: $|x - 2019| = 1000$

Solution: We have $x - 2019 = \pm 1000$, so $x = 2019 \pm 1000 = 1019$ or 3019.

Section B. ***Each correct answer is worth 2 points.***

8. What number, decreased by 2019, is 4 times itself?

Solution: If x is the number, then $x - 2019 = 4x$. Therefore, $3x = -2019$, so $x = -673$.

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9. Find the solution—expressed as an ordered pair (x, y)—to the system $\begin{cases} 3x + 5y = 20 \\ 5x - 6y = 19 \end{cases}$

Solution 1: Solve (say) the first equation for y: $y = \frac{20-3x}{5} = 4 - 0.6x$.
Now substitute into the second equation: $5x - 6(4 - 0.6x) = 19 \implies 8.6x = 43 \implies x = 5$.
Then $y = 4 - 0.6x = 4 - 3 = 1$.

Solution 2: Multiply the first equation by 5 and the second by -3, then add them together, which eliminates the variable x: $(15x + 25y) - (15x - 18y) = 100 - 57 \implies 43y = 43 \implies y = 1$. Then substitute $y = 1$ in either equation, and solve for the value of x, which is 5.

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10. Write an equation for the line passing through the vertices of the ellipse $\dfrac{(x-2018)^2}{9} + \dfrac{(y+2019)^2}{4} = 1$.

Solution: The vertices of an ellipse are the endpoints of its ***major*** axis (that is, the two most curved points on the ellipse). From the given equation, the center of the ellipse is $(2018, -2019)$; the (horizontal) width of the ellipse is $2\sqrt{9} = 6$, and the (vertical) height is $2\sqrt{4} = 2$, so the major axis is horizontal, and the line containing the vertices is $y = -2019$.

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11. Choose (at random) a number from the set of ***negative even two-digit*** integers, and (also at random) a number from the set of ***positive odd three-digit*** integers. What is the probability (expressed as a percent) that the product of their squares is ***odd***?

Solution: If the first number is M and the second is N, we want the probability that $K = M^2N^2$ is odd. But because M is even, M^2 (and therefore K) will always be even; that is, this probability is 0.

Note that the details about (e.g.) "negative even two-digit," and even the fact that we make these selections "at random" (uniformly), are mostly a distraction. The only thing we need to know is that M must be even.

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12. Convert $(5\sqrt{2}, 135°)$ from polar coordinate form (r, θ) to rectangular coordinate form (x, y).

Solution: Polar coordinates and rectangular coordinates are related by $x = r\cos\theta$ and $y = r\sin\theta$. Noting that an angle of $135°$ in standard position has a terminal ray in the second quadrant, we have

$$(x, y) = \left(5\sqrt{2}\cos(135°),\ 5\sqrt{2}\sin(135°)\right) = \left(5\sqrt{2} \cdot \tfrac{-\sqrt{2}}{2},\ 5\sqrt{2} \cdot \tfrac{\sqrt{2}}{2}\right) = (-5, 5).$$

Section C. ***Each correct answer is worth 3 points.***

13. Given: $a + b + c = 14$, $ab = 14$, and $c^2 = a^2 + b^2$. Find the numerical value of c.

Solution 1: One approach to solving this system would be to square the first equation to get $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 196$. On the left side, we can replace $a^2 + b^2$ with c^2, ab with 14, and $2ac + 2bc = 2c(a + b)$ with $2c(14 - c)$. Simplifying, this leaves $196 = 2c^2 + 2(14) + 2c(14 - c) = 28 + 28c$, so $c = 6$.

Solution 2: Adding $28 = 2ab$ to the third equation gives $c^2 + 28 = a^2 + 2ab + b^2 = (a + b)^2$. From the first equation, we know that $a + b = 14 - c$, so $c^2 + 28 = (14 - c)^2 = c^2 - 28c + 196$. Subtracting c^2 from both sides, we are left with $28 = -28c + 196$, so $c = 6$.

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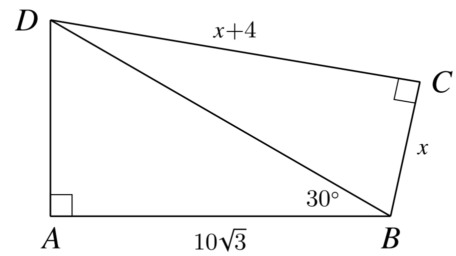
14. For quadrilateral $ABCD$ shown on the right, find the exact value of the perimeter.



Solution: In the figure, $\triangle ABD$ is a 30-60-90 right triangle, so leg $AD = 10$ and hypotenuse $BD = 20$. Then in right $\triangle BCD$, we have $x^2 + (x + 4)^2 = 400$, so $0 = 2x^2 + 8x - 384 = 2(x - 12)(x + 16)$. The only positive solution is $x = 12$—that is, $\triangle BCD$ has side lengths 12, 16, and 20. Therefore, the perimeter is

$$12 + 16 + 10 + 10\sqrt{3} = 38 + 10\sqrt{3}.$$

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15. Simplify, writing your answer in the form $\boldsymbol{a + bi}$: $\dfrac{2020 + 2020i + 2019i^2 + 2019i^3}{1 - i}$

Solution: Noting that $i^2 = -1$ and $i^3 = -1$, the numerator reduces to $2020 + 2020i - 2019 - 2019i = 1 + i$. To complete the simplification, we multiply and divide by $1 + i$, the conjugate of the denominator:

$$\frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+i)^2}{2} = \frac{1+2i+i^2}{2} = i$$

PART II: 30 Minutes; CALCULATORS NEEDED

Section A.* *Each correct answer is worth 1 point.

1. The area of a rectangle, with integer sides, is 2019 square centimeters. Find ***all possible integral values*** for its perimeter.

Solution: 2019 only has four factors: 1, 3, 673, and 2019. So, this rectangle is either 1×2019 (perimeter 4040 cm) or 3×673 (perimeter 1352 cm).

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2. The average of 19 numbers is 20. When one number is dropped, the average of the remaining set of numbers is 19. What number was dropped?

Solution: If S is the sum of the 19 numbers, and x is the dropped number, then $\frac{S}{19} = 20$—meaning $S = 380$—and $\frac{S-x}{18} = \frac{380-x}{18} = 19$, so $380 - x = 342$, and $x = 38$.

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3. For ***how many*** integral values of x does there exist a triangle whose sides have length 6, 8, and x?

Solution: Recall that a, b, c can be the side lengths of a triangle if all three of the so-called "triangle inequalities" hold:

$$a < b + c, \quad b < a + c, \quad c < a + b$$

So we need $6 < x + 8$, $8 < x + 6$, and $x < 14$. The first of these is always true (for any positive x), while the second and third imply that we need an integer x properly between 2 and 14; that is, $3, 4, 5, \ldots, 13$, which is 11 possible values.

4. Express as a ratio in simplest form: $\dfrac{\text{19 hours, 30 minutes}}{\text{20 hours, 30 minutes}}$

Solution: To reduce this ratio, express all parts with the same units—either $\frac{19.5 \text{ hours}}{20.5 \text{ hours}}$ or $\frac{1170 \text{ minutes}}{1230 \text{ minutes}}$. (Of course, if you want to make your life more difficult, you could express everything in units of days, or seconds, or something else.) In any case, the units cancel, and the numbers simplify to $\frac{39}{41}$.

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5. Subtract and express your answer using Roman numerals: MMXIX – CCCLIII.

Solution: MMXIX is (surprise!) 2019, and CCCLIII is 353, so the difference is 1666. Note that this equals $1000 + 500 + 100 + 50 + 10 + 5 + 1$, so to express this in Roman numerals, we need exactly one of each of the standard letters: MDCLXVI.

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6. Write the exact value of $\left|\sqrt{2} - \pi\right|$.

Solution: Subtract the smaller number from the larger one: $\left|\sqrt{2} - \pi\right| = \pi - \sqrt{2}$.

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7. Over a number of years, the Beavers played 250 games and won 8 more games than they lost. Find the percent of the games they won. Express your answer to the nearest tenth of one percent.

Solution: They won W games and lost L games, where $W = L + 8$ and $W + L = 250$. Therefore, $W = 129$ (and $L = 121$), and the percent won is $\frac{W}{250} = 51.6\%$ (exactly—no rounding needed).

Section B*. *Each correct answer is worth 2 points.

8. What is the ***exact*** area of a triangle whose sides are $19, 20,$ and 19?

Solution 1: We note that this is an isosceles triangle with base length $b = 20$. The altitude to the base (length h) forms two right triangles, with legs 10 and h, and hypotenuse 19, so $h = \sqrt{19^2 - 10^2} = \sqrt{261} = 3\sqrt{29}$, and the area is $\frac{1}{2}bh = 10\sqrt{261} = 30\sqrt{29}$ square units.

Solution 2: We can also find the area using Hero's (Heron's) formula, valid for any triangle: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the side lengths, and $s = \frac{1}{2}(a + b + c)$ is the semiperimeter. For this triangle, $s = 29$, so $A = \sqrt{29 \cdot 10 \cdot 9 \cdot 10} = 30\sqrt{29}$.

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9. Assuming all angles acute, solve for θ (in degrees): $\sin(20°) = \cos(2\theta)$.

Solution: For any angle α (acute or not), $\sin(\alpha) = \cos(90° - \alpha)$, so $2\theta = 70°$, and $\theta = 35°$.

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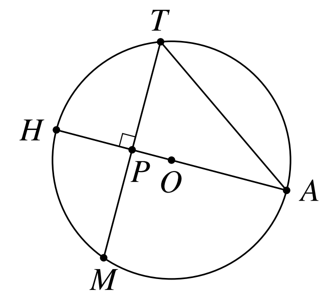
10. You are looking for integers n such that both $n + 100$ and $n + 164$ are perfect squares. One such integer is $n = 125$, because $125 + 100 = 225 = 15^2$ and $125 + 164 = 289 = 17^2$. There are two other integers n less than 125 with this property. Find ***one*** of those integers.

Solution: While a trial and error approach would work, we can get to the answers more quickly by noting that if $n + 100 = k^2$ and $n + 164 = j^2$ (with $j > k$), then $n = k^2 - 100 = j^2 - 164$. Therefore, $j^2 - k^2 = (j - k)(j + k) = 64$. Considering the factors of 64, we could have:

- $j - k = 1$ and $j + k = 64 \implies j = \frac{65}{2}$ and $k = \frac{63}{2}$ (not integers).
- $j - k = 2$ and $j + k = 32 \implies j = 17$ and $k = 15$ ($n = 125$, the given solution).
- $j - k = 4$ and $j + k = 16 \implies j = 10$ and $k = 6$ ($n = -64$).
- $j - k = 8$ and $j + k = 8 \implies j = 8$ and $k = 0$ ($n = -100$).

11. Given circle O as shown, with $HP = 4$ and $TM = 16$. Find the value of TA, correct to the nearest hundredth.



Solution: When a radius (like $\overline{OH}$) is perpendicular to a chord (like $\overline{MT}$), it bisects that chord, so $MP = PT = 8$. By the intersecting chord theorem, $HP \cdot PA = MP \cdot PT = 64$, so $AP = 16$ and $TA = \sqrt{8^2 + 16^2} = \sqrt{320} \doteq 17.89$.

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12. The product of two ***consecutive negative even integers*** is 5 more than 2019. Find the ***sum*** of these two integers.

Solution: Let n and $n+2$ be the two integers; then $n(n+2) = 2024$. The negative solution to this quadratic equation is $n = -46$, so $n + (n + 2) = -90$.

Note we can solve the quadratic by factoring into $(n + 46)(n - 44) = 0$, or using the quadratic formula, or by noting that $2024 = 45^2 - 1 = (45 + 1)(45 - 1)$.

Section C. ***Each correct answer is worth 3 points.***

13. Flying with a steady tailwind, a plane flies 450 miles from airport M to airport N in $2\frac{1}{2}$ hours. With the wind remaining steady as before, the return trip to airport M takes 3 hours, 20 minutes. What is the wind speed?

Solution: If w is the wind speed, and p the speed of the plane, we have $(p+w)\left(2\frac{1}{2}\right) = 450 = (p-w)\left(3\frac{1}{3}\right)$. Rearranging, we have $p + w = 180$ and $p - w = 135$; subtracting these equations, we have $2w = 45$, so $w = 22.5$ mph.

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14. Factor completely over the rational numbers: $a^4 - 121 - 16a^3 + 64a^2$

Solution: $a^4 - 121 - 16a^3 + 64a^2 = a^2(a^2 - 16a + 64) - 121 = a^2(a - 8)^2 - 11^2$. This is a difference of squares, so it can be written as $(a^2 - 8a + 11)(a^2 - 8a - 11)$.

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15. Basketball player Dakota currently has made $\boldsymbol{A}$ free throws in $\boldsymbol{B}$ attempts, and his/her free throw shooting percentage is less than 50%. How many consecutive free throws must Dakota make in order to raise her/his average to exactly 50%? Express your answer in terms of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Solution: If Dakota makes x consecutive free throws to get to 50%, we must have $\frac{A+x}{B+x} = \frac{1}{2}$, so $2(A + x) = B + x$. Solving for x gives us $x = B - 2A$.