BLUFFTON UNIVERSITY

MATHEMATICS CONTEST 2018
ANSWER KEY

PART I

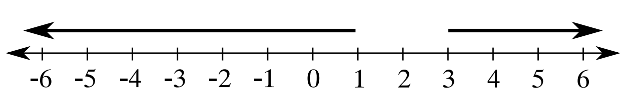
Section A (1 point each)

1. 4 ***or*** four
2. $x = -1$
3. 59 and 61
4. 10% decrease
5. $\frac{2}{101}$
6. 12 and 13
7. $b = 6$

Section B (2 points each)

8. $\sqrt{181}$ or $\sqrt{19}$. ***(both)***
9. 60 or 300 [degrees] ***(both)***
10. 120 or 280 or 320 or 480 [miles] ***(all)***
11. no solution (none, impossible, $\varnothing$, . . .)
12. -38 or 22 ***(both)***

Section C (3 points each)

13. $b = 1$
14. $36 + 18\sqrt{3}$ square units
15. 

PART II

Section A (1 point each)

1. 0 ***or*** (any negative integer)
2. $(0.45, -4.05)$ ***or*** $(\frac{9}{20}, -4\frac{1}{20})$
3. \$18.20
4. $x = 20.18$ (degrees)
5. 18
6. $x = 20$
7. 6014

Section B (2 points each)

8. 6.72 ***or*** $6\frac{18}{25}$ units
9. 4 mph
10. 3742_{eight}
11. -11
12. $\frac{1}{2}$

Section C (3 points each)

13. $h = 4r$
14. mode = 2000
15. $n = 8$

PART I: 30 Minutes; NO CALCULATORS

Section A. ***Each correct answer is worth 1 point.***

1. ***How many*** positive integral factors does 2018 have?

Solution: The prime factorization of 2018 is $2 \cdot 1009$, so it has FOUR factors: $1, 2, 1009$, and 2018.

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2. Solve for x: $20x + 20 = 18x + 18$

Solution: Collect like terms: $2x = -2$, so $x = -1$.

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3. ***Twin primes*** (sometimes called pairs of primes) are prime numbers that differ by two, such as 3 and 5, 11 and 13, etc. List all twin primes between 50 and 70.

Solution: There are only four primes in that range (in particular, note that $51 = 3 \cdot 17$ is ***not*** a prime); the one twin-prime pair is 59 and 61.

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4. Find the percent of decrease from 20 to 18.

Solution: Percent change is computed as (size of change) / (original value), so we have $\frac{2}{20} = 0.1 = 10\%$.

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5. Find $0.\overline{2018} - 0.\overline{1820}$ and express as a fraction (that is, a ratio of two integers) in simplest form.

Solution: A repeating decimal $0.\overline{pqrs}$ is equal to the fraction $\frac{pqrs}{9999}$, so we have $0.\overline{2018} - 0.\overline{1820} = \frac{2018-1820}{9999} = \frac{198}{9999} = \frac{2 \cdot 99}{101 \cdot 99} = \frac{2}{101}$.

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6. The cube root of 2018 (that is, $\sqrt[3]{2018}$) lies between what two positive integers?

Solution: By trial and error, we find that $12^3 = 1728$ and $13^3 = 2197$, so $\sqrt[3]{2018}$ falls between 12 and 13.

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7. The six-digit number $62a11b$ is divisible by 9 ***and*** divisible by 11. What is the value of b?

Solution: A number is divisible by 9 if the sum of its digits is divisible by 9, so we need $6+2+a+1+1+b = 10 + a + b$ to be divisible by 9, so either (1) $a + b = 8$ or (2) $a + b = 17$.

In order to be divisible by 11, we must obtain a multiple of 11 when we alternately add and subtract the digits: $6 - 2 + a - 1 + 1 - b = 4 + a - b$, which is divisible by 11 if (3) $a - b = -4$ or (4) $a - b = 7$.

Note that if $a + b = x$ and $a - b = y$, then $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$. Because a and b must be single-digit integers, we need $x + y$ (and $x - y$) to be even, and $x + y < 20$, which means that only (1) and (3) given a valid answer: $a = \frac{8-4}{2} = 2$ and $b = \frac{8+4}{2} = 6$.

Alternatively, we can eliminate equation (2), for which the possible solutions (either $a = 8$ and $b = 9$, or vice versa) are not compatible with (3) or (4). Likewise, equation (4) implies that either a is 7, 8, or 9, and b is 0, 1, or 2, all of which would fail to satisfy (1). That leaves only equations (1) and (3), which (as we noted above) yield the solution $a = 2$ and $b = 6$.

Section B. Each correct answer is worth 2 points.

8. Two sides of a right triangle have lengths of 9 and 10. Find the exact length of the third side.

Solution: There are two possible answers: $\sqrt{181}$ or $\sqrt{19}$ (both were needed to get credit for this problem). The two given side lengths might be the ***legs*** of the triangle, making the hypotenuse $\sqrt{9^2 + 10^2} = \sqrt{181}$. ***Or*** the longer side might be the ***hypotenuse,*** meaning that the other leg has length $\sqrt{10^2 - 9^2} = \sqrt{19}$.

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9. Solve for θ (in degrees), with $0° < \theta < 360°$, if $\cos\theta = \frac{1}{2}$.

Solution: In a right triangle, the cosine is the adjacent-to-hypotenuse ratio, so a reference triangle for $\cos\theta = \frac{1}{2}$ could have a hypotenuse of length 2 and an adjacent leg of length 1—which would be a 30-60-90 right triangle, with $\theta = 60°$. The cosine function takes on the value $\frac{1}{2}$ one other time in the interval $(0°, 360°)$, when $\theta = 300°$. (That is, that reference triangle could be placed in either the first quadrant, or the fourth quadrant.)

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10. Two trains A and B are on a straight track and are 300 miles apart. Train A is going 40 mph and Train B is going 50 mph. If they continue to travel on that track at those speeds, how many miles apart will they be after 2 hours?

Solution: There are FOUR possible answers, depending on the relative directions that the two trains are traveling. For simplicity, assume A is west of B.

(1) A→ ←B: If A is going east and B is going west (toward each other), they will be 120 miles apart.
(2) ←A ←B: If both go west (B is "chasing" A), they will be 280 miles apart.
(3) A→ B→: If both go east (A is chasing B), they will be 320 miles apart.
(4) ←A B→: If A is going west and B is going east (opposite directions), they will be 480 miles apart.

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11. Solve for x: $3x^2 - 5x + 1820 = 2018 - 5x + 3(x^2 + 7)$

Solution: After collecting like terms, we find that all terms involving x cancel out, and the equation reduces to $1820 = 2039$, so there are no solutions. Note: We accepted "no solutions" or "impossible" or "none" or "∅," but we did not accept "0" (which implies that 0 is a solution).

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12. If $x^2 - x - 20 = -18$, find the value of $20x - 18$.

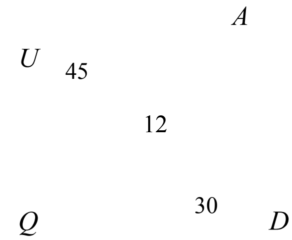
Solution: When written in standard form, we see that the given equation factors into $(x + 1)(x - 2) = 0$, so the solutions $x = -1$ or $x = 2$. Therefore, $20x - 18$ is either -38 or 22.

Section C. Each correct answer is worth 3 points.

13. Given that $3, a, b, c, 0$ is a Fibonacci-style sequence, find the value of b.

Solution: In a Fibonacci-style sequence, we add two consecutive terms to find the next, so $3 + a = b$, $a + b = c$, and $b + c = 0$. The third equation implies that $c = -b$. Together with the second equation, this implies that $a = -2b$. Finally, with the first equation, we have $3 - 2b = b$, so $b = 1$.

14. Find the ***exact*** area of quadrilateral $QUAD$ on the right.



<u>**Solution:**</u> $\triangle QUD$ is a 30-60-90 right triangle with hypotenuse 12, so the legs are $QU = 6$ and $QD = 6\sqrt{3}$, and the area is $18\sqrt{3}$ square units. Meanwhile, isosceles right $\triangle UAD$ has legs $6\sqrt{2}$, so the area is 36 square units, making the total $36 + 18\sqrt{3}$ square units.

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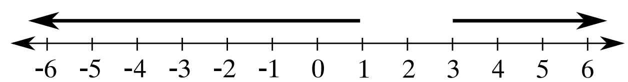
15. Solve the inequality $|x - 2| > 1$, and graph its solution on the given number line.

<u>**Solution:**</u> If $|x - 2| > 1$, then either $x - 2 > 1$ or $x - 2 < -1$, or equivalently, $x > 3$ or $x < 1$.



PART II: 30 Minutes; CALCULATORS NEEDED

<u>Section A.</u> Each correct answer is worth 1 point.

1. Give an example of an integer which is ***not*** a natural number.

<u>**Solution:**</u> A natural number is a positive integer, so any non-positive integer will do $(0, -1, -2, \text{etc.})$.

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2. Find the vertex of the quadratic function $y = 20x^2 - 18x$. Express as the ordered pair (x, y), where x and y are written to the nearest hundredth.

<u>**Solution 1:**</u> Complete the square: $y = 20(x^2 - 0.9x + 0.45^2) - 20 \cdot 0.45^2 = 20(x - 0.45)^2 - 4.05$. The form $y = a(x - h)^2 + k$ is sometimes called "vertex form," because we can read the vertex (h, k) directly out of the equation: $(0.45, -4.05)$.

<u>**Solution 2:**</u> Using calculus, we find the derivative $y' = 40x - 18$. This equals 0 when $x = \frac{18}{40} = 0.45$, which gives us the location of the horizontal tangent (and hence the first coordinate of the vertex). The second coordinate is $f(0.45) = 20 \cdot 0.45^2 - 18 \cdot 0.45 = -4.05$.

Note: We also accepted the answer in (reduced) fractional form—that is, $(\frac{9}{20}, -\frac{81}{20}$ or $-4\frac{1}{20})$.

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3. J. Denny Beaver bought an item and paid \$20.18, including sales tax of 10.9%. What was the cost of the item before the sales tax?

<u>**Solution:**</u> The price p must satisfy $1.109p \approx 20.18$ (equal after rounding to the nearest cent), so $p \approx \frac{20.18}{1.109} \doteq 18.1965\ldots$. Checking \$18.20 confirms that this value produces a final bill of \$20.18.

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4. Find the value of x in degrees, where x is an acute angle, if $\cos^2(x) = 1 - \sin^2(20.18°)$.

<u>**Solution:**</u> Because $\cos^2 x + \sin^2 x = 1$ (for any value of x), the simplest answer—and the only acute angle that works—is $x = 20.18°$.

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5. Only one positive integer is exactly twice the sum of its digits. Find this two-digit number.

<u>**Solution:**</u> The two-digit number $10a + b$ must be equal to $2(a + b)$, so $8a = b$. The only possible solution (among single-digit values for a and b, with $a \neq 0$) is $a = 1, b = 8$.

6. Solve for x: $20^{18/(x-2)} = 20$.

Solution: The equation $a^t = a$ (for $a > 0$ and $a \neq 1$) implies that the exponent t equals 1. In this case, $t = \frac{18}{x-2} = 1$, so $18 = x - 2$, and therefore $x = 20$.

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7. $f(x) = x^3 - 20x^2 + 18x + 2018$. If the domain is $\{-1, 0, 1\}$, find the ***sum*** of all the values of the range.

Solution: The range consists of the values of $f(x)$ for all values in the domain, so we have

$$f(-1) = -1 - 20 - 18 + 2018 = 1979$$
$$f(0) = 2018$$
$$f(1) = 1 - 20 + 18 + 2018 = 2017$$

and the sum of these values is 6014.

Section B. ***Each correct answer is worth 2 points.***

8. The two legs of a right triangle have lengths 7 and 24. Find the exact length of the altitude going to the hypotenuse.

Solution: The hypotenuse of that triangle is $\sqrt{7^2 + 24^2} = 25$. Viewing the legs as the base and altitude, the area is $\frac{1}{2}(7)(24) = 84$ square units. On the other hand, with the hypotenuse as the base, the area is $\frac{1}{2}(25)(h)$; equating these gives $h = 6\frac{18}{25} = 6.72$ units.

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9. Jenny Beaver's jogging path is an equiangular triangle with vertices X, Y, and Z. She jogs at a constant speed of 3 mph from X to Y, a constant speed of 4 mph from Y to Z, and a constant speed of 6 mph from Z back to X. What is her average speed of the entire round trip?

Solution: Note that an equiangular triangle is also equilateral. If the side length of the triangle is s (miles), then the three parts of Jenny's jog take $\frac{1}{3}s + \frac{1}{4}s + \frac{1}{6}s = \frac{3}{4}s$ hours, and her average speed is $\frac{\text{distance}}{\text{time}} = \frac{3s}{3s/4} = 4$ mph. (This is called the ***harmonic mean*** of 3, 4, and 6.)

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10. Express 2018 in ***base 8***. For example, 29 is 35_{eight} because $3 \cdot 8^1 + 5 \cdot 8^0 = 29$, and 375 is 567_{eight} because $5 \cdot 8^2 + 6 \cdot 8^1 + 7 \cdot 8^0 = 375$.

Solution: The powers of 8 are 1, 8, 64, 512, 4096, etc. We divide by those powers, starting with 512 (the highest power that goes into 2018):

$$2018 \div 512 = 3 \text{ with a remainder of } 482$$
$$482 \div 64 = 7 \text{ with a remainder of } 34$$
$$34 \div 8 = 4 \text{ with a remainder of } 2$$

So $2018 = 3742_{\text{eight}}$.

11. If $2x^3 - 0x^2 + 1x - 8$ is divided by $x + 1$, what is the remainder?

Solution 1: By the remainder theorem, the remainder when polynomial $f(x)$ is divided by $x - c$ is $f(c)$, so in this case the remainder would be $2(-1)^3 - 0(-1)^2 + 1(-1) - 8 = -2 - 1 - 8 = -11$.

Solution 2: Using synthetic division, we find

$$\begin{array}{r|rrrr} -1 & 2 & 0 & 1 & -8 \\ & & -2 & 2 & -3 \\ \hline & 2 & -2 & 3 & -11 \end{array}$$

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12. Simplify and express in numerical form: $\log_{20} 20 + \log_{18} 1 + \log_3 9 - \log_2 8 + \log_4 2$.

Solution: $\log_{20} 20 + \log_{18} 1 + \log_3 9 - \log_2 8 + \log_4 2 = 1 + 0 + 2 - 3 + \frac{1}{2} = \frac{1}{2}$.

***Section C.* Each correct answer is worth 3 points.**

13. A right circular cone has the same volume as a sphere. If the base of the cone has the same radius as the sphere, find the height of the cone in terms of that radius r.

Solution: The volume of a cone is $\frac{1}{3}\pi r^2 h$, and the volume of a sphere is $\frac{4}{3}\pi r^3$, so if these are to be equal, we must have $h = 4r$.

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14. A set of five natural numbers has a mean, median, and mode. The mean is 2018, and the median is 2 more than the mode. The two non-median and non-mode numbers are 2018 and 2070. Write the value of the mode.

Solution: Let A and $A + 2$ be the values of the mode and median. Note that in order to have a mode, A must appear (at least) twice in the list, and $A + 2$ must be the third number, so the five numbers are $A, A, A + 2, 2018, 2070$. With a mean of 2018, we must have $3A + 4090 = 5 \cdot 2018 = 10090$, so $3A = 6000$, and therefore the mode is 2000.

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15. Find the value of n if $\displaystyle\sum_{k=1}^{n} (3k - 1) = 100$.

Solution 1: The formula for the sum of consecutive integers is $1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{1}{2}n(n + 1)$, so $\sum_{k=1}^{n}(3k - 1) = 3\sum_{k=1}^{n} k - n = \frac{3}{2}n(n+1) - n = \frac{1}{2}(3n^2 + n)$. This is equal to 100 when $3n^2 + n = 200$, for which the only integral solution is $n = 8$.

Solution 2: Expand the sum like so:

$$\sum_{k=1}^{n} (3k - 1) = \overbrace{(3 \cdot 1 - 1)}^{k=1} + \overbrace{(3 \cdot 2 - 1)}^{k=2} + \overbrace{(3 \cdot 3 - 1)}^{k=3} + \cdots + \overbrace{(3 \cdot n - 1)}^{k=n} = 2 + 5 + 8 + \cdots + (3n - 1)$$

By trial and error, we can determine that we need 8 terms in order to reach 100.