BLUFFTON UNIVERSITY

MATHEMATICS CONTEST 2017
ANSWER KEY

PART I

Section A

1. 338
2. 3 days
3. 113
4. 20%
5. -22 *or* $(0, -22)$
6. $n = 100$
7. 5

Section B

8. 1
9. $3\sqrt{3}$ square units
10. $x = 4$
11. $a + b = 34$
12. 2584

Section C

13. $a = 20, b = 17, c = -1$
14. $9\frac{7}{18}$
15. $17 + 20i$

PART II

Section A

1. A (B) C D E
2. perimeter
3. 433
4. 95
5. 65° *or* 5°
6. 29 days
7. 261 women

Section B

8. 0 or none
9. 51 mph
10. 2015
11. 17
12. 328.41 square units

Section C

13. 14 sides
14. $\dfrac{2b^5d^2}{a^7c^6}$
15. 1855 units

BLUFFTON UNIVERSITY

MATHEMATICS CONTEST 2017
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Simplify: $20 \times 17 - 2^0 - 1^7$

Solution: $20 \times 17 - 2^0 - 1^7 = 340 - 1 - 1 = \underline{338}$.

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2. In a month of 31 days, how many different days of the week occur 5 times during that month?

Solution: After 28 days, each day has occurred four times; the last 3 days of the month occur five times.

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3. What is the smallest 3-digit prime number that contains only odd digits?

Solution: The first 3-digit number containing only odd digits is not prime ($111 = 3 \cdot 37$). However, 113 is prime.

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4. What is the percentage of *multiples of* 4 that *end in* 4?

Solution: The final digit of the multiples of 4 cycles through the sequence 4, 8, 2, 6, 0, so one in five $= \underline{20\%}$ of all multiples of 4 end in 4.

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5. Find the y-intercept of the line $x - 20 = 4(y + 17)$.

Solution 1: The y-intercept has $x = 0$, so we need to solve $-20 = 4(y + 17)$. Therefore, $-5 = y + 17$, so $y = \underline{-22}$.

Solution 2: We can rearrange the equation to get slope-intercept form: $y = \frac{1}{4}x - 22$. Therefore, the y-intercept is $(0, -22)$.

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6. If 2017 is divided by the integer n, the quotient is 20 and the remainder is 17. What is the value of n?

Solution: If a/b has quotient q and remainder r, then $a = bq + r$. In this instance, $2017 = 20n + 17$, so clearly $n = \underline{100}$.

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7. Evaluate and express your answer in simplest form (without !): $\dfrac{4!\,7!\,10!}{9!\,8!\,3!}$

Solution: Recall that $n! = n(n-1)(n-2)\cdots(2)(1)$, and more specifically, $\frac{(n+1)!}{n!} = n + 1$. Therefore, $\dfrac{4!\,7!\,10!}{9!\,8!\,3!} = \dfrac{4!}{3!} \cdot \dfrac{10!}{9!} \cdot \dfrac{7!}{8!} = \dfrac{4 \cdot 10}{8} = \underline{5}$.

Section B. Each correct answer is worth 2 points.

8. The equation $x^2 = 1$ has two solutions. The equation $x^2 = x$ also has two solutions. Find the sum of all four of these solutions.

Solution: These solution sets can be found either by factoring, the quadratic equation, (for the first) extracting square roots, or (for the second) dividing both sides by x. The solutions for $x^2 = 1$ are -1 and 1, and the solutions for $x^2 = x$ are 0 and 1, so the four solutions sum to $\underline{1}$.

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9. The length of each side of an equilateral triangle is $2\sqrt{3}$. Find the exact area of that triangle.

Solution 1: Drop an altitude from any vertex; this altitude is then the shared leg of two right triangles with hypotenuse $2\sqrt{3}$ and leg $\sqrt{3}$. By the Pythagorean Theorem, the altitude's length is therefore $\sqrt{(2\sqrt{3})^2 - (\sqrt{3})^2} = \sqrt{12-3} = \sqrt{9} = 3$, so the area is $\frac{1}{2}bh = \frac{1}{2}(2\sqrt{3})(3) = 3\sqrt{3}$.

Solution 2: Heron's (or Hero's) formula says that the area of a triangle is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter $(a+b+c)/2$. In this case, $a = b = c = 2\sqrt{3}$, so $s = 3\sqrt{3}$, and the area is $\sqrt{3\sqrt{3} \cdot (\sqrt{3})^3} = \sqrt{27} = 3\sqrt{3}$.

Solution 3: The area of an equilateral triangle with side length s is $A = \frac{1}{4}s^2\sqrt{3} = 3\sqrt{3}$.

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10. If $\begin{bmatrix} y & -1 \\ 1 & x \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 17 \end{bmatrix} = \begin{bmatrix} 23 \\ 88 \end{bmatrix}$, find the value of x.

Solution: Multiplying out the matrices leads to the two equations $20y - 17 = 23$ and $20 + 17x = 88$. Because we only were asked for the value of x, we can completely ignore the first equation; solving the second gives $17x = 68$, so $x = \underline{4}$.

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11. The sequence $20, a, 17, b, \ldots$ is an arithmetic progression. Find the value of $a + b$.

Solution 1: In an arithmetic progression, consecutive items differ by the same constant, so a must be 18.5, and b must be 15.5, and $a + b = 34$.

Solution 2: Each term in an arithmetic progression is the average (arithmetic mean) of its two neighboring terms, so $\frac{a+b}{2} = 17$, and therefore $a + b = 34$. Note: From this method, we can see that we don't need to know the first term in the sequence (20).

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12. In the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, each term (starting with the third term) is found by adding the two previous terms ($1+1 = 2, 1+2 = 3, 2+3 = 5$, etc.) What is the value of the <u>first term</u> in this sequence that is larger than 2017?

Solution 1: With care and patience, we find: $\ldots, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \underline{2584}$ (the 18th Fibonacci number).

Solution 2: While not really helpful without a calculator, here is an approach for solving for n in the more general inequality $f_n > K$, where f_n is the nth Fibonacci number. Use the fact that f_n is equal to $\phi^n/\sqrt{5}$ rounded to the nearest integer, where $\phi = \frac{\sqrt{5}+1}{2}$ is the Golden ratio. Solving $\phi^n > K\sqrt{5}$ gives $n > \log_\phi(K\sqrt{5})$. With $K = 2017$, we need n larger than about 17.5, which (again) points us to the 18th Fibonacci number.

Section C. *Each correct answer is worth 3 points*.

13. Solve this system of equations: $\begin{cases} a + b + c = 36 \\ 2a - c = 41 \\ 3b + c = 50 \end{cases}$

Solution: We can reduce to the two-variable system $\begin{cases} 3a + b = 77 \\ 2a + 3b = 91 \end{cases}$ by adding the second equation to the first and third. From the first of these, we have $b = 77 - 3a$, so $3b = 231 - 9a$. Substituting in the other equation gives $-7a = -140$, so $a = 20$. Therefore, $b = 77 - 3a = 17$, and $c = -1$.

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14. Find the number that is $\frac{2}{3}$ the distance from $7\frac{2}{3}$ to $10\frac{1}{4}$. Express your answer as a mixed number in simplest form.

Solution 1: The distance from $7\frac{2}{3}$ to $10\frac{1}{4}$ is $10\frac{1}{4} - 7\frac{2}{3} = 10\frac{3}{12} - 7\frac{8}{12} = 2\frac{7}{12} = \frac{31}{12}$. So we want

$$7\tfrac{2}{3} + \tfrac{2}{3} \cdot \tfrac{31}{12} = 7\tfrac{12}{18} + \tfrac{31}{18} = 7\tfrac{43}{18} = 9\tfrac{7}{18}.$$

Solution 2: Given two numbers x and y, the number which is located at the ratio $\frac{b}{a+b}$ from x to y—or equivalently, $\frac{a}{a+b}$ from y to x—is $\frac{ax+by}{a+b}$. So with $x = 7\frac{2}{3} = \frac{23}{3}$ and $y = 10\frac{1}{4} = \frac{41}{4}$, the number we are seeking is

$$\frac{x + 2y}{3} = \frac{\frac{23}{3} + \frac{41}{2}}{3} = \frac{\frac{46}{6} + \frac{123}{6}}{3} = \frac{46+123}{18} = 9\frac{7}{18}$$

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15. Simplify: $\frac{20-17i}{3+i} + 12.7 + 27.1i$

Solution: Multiply and divide the first expression by the conjugate of the denominator:

$$\frac{20 - 17i}{3 + i} = \frac{(20 - 17i)(3 - i)}{(3 + i)(3 - i)} = \frac{60 - 20i - 51i - 17}{9 + 1} = 4.3 - 7.1i$$

Adding the other two terms gives $(4.3 + 12.7) + (-7.1 + 27.1)i = 17 + 20i$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. *Each correct answer is worth 1 point*.

1. On your answer sheet, circle the letter of the one set which is not equal to the other four.
   A) counting numbers B) whole numbers C) $\{1, 2, 3, 4, \ldots\}$
   D) natural numbers E) positive integers

Solution: The whole numbers (B) include 0, unlike all of the others.

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2. What name is given to the sum of the lengths of the sides of a polygon?

Solution: The sum of the lengths of the sides of a polygon is called the perimeter.

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3. Four consecutive odd integers have a sum of 1720. What is the largest of these four integers?

Solution: If n is the largest of these integers, then we have $(n-6)+(n-4)+(n-2)+n = 4n-12 = 1720$. Then $4n = 1732$, so $n = 433$.

4. Dakota's test scores are $75, 86, 82$, and 77. On the next test, Dakota brought her average up 3 points. What score did Dakota get on that fifth test?

Solution 1: Her first four test scores sum to $75 + 86 + 82 + 77 = 320$, for an average of 80. If she raised her average by 3 points, her new total must be $5 \cdot 83 = 415$, so she scored 95 points on the fifth test.

Solution 2: In general, if the score on her nth test raises her average by k points, that means she scored nk points more than her previous average. (This observation helps us to see why, as you take more tests and quizzes, it becomes increasingly harder to raise your average.)

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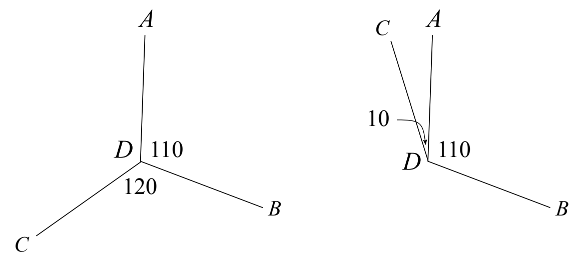
5. Points A, B, and C lie on a circle. The measure of $\overset{\frown}{AB}$ is $110°$, and the measure of $\overset{\frown}{BC}$ is $120°$. Find the measure of $\angle ABC$.



Solution: The points A, B, and C can fall on the circle in two different ways, as shown on the right. $\angle ABC$ is an *inscribed* angle, meaning its vertex is on the circle. The measure of such an angle is half the measure of the arc that it *subtends*—that is, $\overset{\frown}{AC}$. In the figure, the measure of $\overset{\frown}{AB}$ is the same as $m\angle ADB = 110°$, while $m\,\overset{\frown}{BC} = m\angle BDC = 120°$. Therefore, $m\,\overset{\frown}{AC}$ is either $130°$ (left case) or $10°$ (right case), so that $m\angle ABC = 65°$ or $5°$.

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6. A water lily doubles itself in size every day. From the time the original plant was placed in a pond until the surface was completely covered took 30 days. How many days did it take for the pond to be half-covered?

Solution: If the pond was covered after 30 days, then it was half-covered one day earlier, on day 29.

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7. There are 435 students at Flubbton University. If the ratio of men to women is 2 to 3, how many women are at Flubbton?

Solution: The fraction of women is $\frac{3}{5}$, and there are $\frac{3}{5} \cdot 435 = 261$ women.

Section B. Each correct answer is worth 2 points.

8. How many of these expressions are equal to one-half of 4^{432}?

   A) 2^{432} B) 4^{216} C) 8^{288} D) 16^{216} E) $\frac{1}{2} \cdot 432^4$

Solution: One-half of 4^{432} can be written as $\frac{1}{2} \cdot (2^2)^{432} = 2^{863}$. This obviously differs from A, and we can also see that it differs from the other expressions: $4^{216} = 2^{432}$, $8^{288} = 2^{864}$, $16^{216} = 2^{864}$, and $\frac{1}{2} \cdot 432^4$ cannot be expressed as a power of 2.

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9. Shelli drove 20 miles from home to the Bluffton University Mathematics Contest in 30 minutes. To the nearest mph, what speed must she average *on her drive home* in order to average 45 mph for the *entire trip* to and from Bluffton?

Solution: If Shelli's entire trip takes t hours, her average speed is $\frac{40}{t}$ mph. In order to have $\frac{40}{t} = 45$, we need $t = \frac{40}{45} = \frac{8}{9}$ hours. The trip to Bluffton took $\frac{1}{2}$ hour, so her trip home takes $t - \frac{1}{2} = \frac{7}{18}$ hour. Therefore, her average speed on the way home was $\frac{20 \text{ miles}}{7/18 \text{ hour}} = \frac{360}{7}$ mph $= 51\frac{3}{7}$ mph—or (to the nearest whole number) 51 mph. *Note:* Her average speed for the whole trip is the *harmonic* mean of 40 mph (average speed going to the contest) and $51\frac{3}{7}$ mph. The harmonic mean of two numbers x and y is $\frac{2}{\frac{1}{x}+\frac{1}{y}}$.

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10. Define the operation $\circledast$ to mean $a \circledast b = a^2 + 4ab + b^2 - 2b$. Find the value of $20 \circledast 17$.

Solution: $20 \circledast 17 = 20^2 + 4(20)(17) + 17^2 - 2(17) = 400 + 1360 + 289 - 34 = \underline{2015}$.

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11. On a multiple-choice test, your grade is determined in this manner: Add 10 points for each question that you answer correctly, subtract 5 points for each question that you answer incorrectly, and you get 0 points for each question left unanswered. If there are 20 questions on the test, and you answered all of them and received a score of 155, how many correct answers did you have?

Solution: If you answer x questions correctly and y incorrectly, then $x + y = 20$ and $10x - 5y = 155$. Then $10x - 5(20 - x) = 155$, so $x = \underline{17}$ correct answers.

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12. One angle of a parallelogram contains $75°$. If the adjacent sides are 20 and 17, find the area of the parallelogram to the nearest hundredth.

Solution: A parallelogram can be divided into two congruent triangles, each with area $\frac{1}{2}ab\sin\theta$, so the area of the parallelogram is $(20)(17)\sin 75° \doteq \underline{328.41}$ square units.

Section C. Each correct answer is worth 3 points.

13. In a certain polygon, the sum of all the interior angles except one is $2017°$. How many sides are in this polygon?

Solution: The interior angles of an n-gon sum to $180(n-2)°$, so $180(n-2)° = 2017° + M°$, where $M°$ is the missing angle. Therefore, $n = 2 + \frac{2017+M}{180} \doteq 13.2 + \frac{M}{180}$. Of course, n is a whole number and $0 < \frac{M}{180°} < 1$, so n must be $\underline{14}$.

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14. Express in simplest form without any negative exponents: $\dfrac{(4a^{-2}b)^2c^{-3}d^{n+1}}{8a^3(b^{-1}c)^3d^{n-1}}$

Solution: $\dfrac{(4a^{-2}b)^2c^{-3}d^{n+1}}{8a^3(b^{-1}c)^3d^{n-1}} = \dfrac{16a^{-4}b^2c^{-3}d^{n+1}}{8a^3b^{-3}c^3d^{n-1}} = \dfrac{2b^5d^2}{a^7c^6}$

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15. In a right triangle, 2017 is the length of the hypotenuse. If the difference in length between the legs is 1063, find the length of the longer leg.

Solution: If the longer leg has length x, then $(x - 1063)^2 + x^2 = 2017^2$. Expanding and collecting like terms gives

$$2x^2 - 2126x - 2\,938\,320 = 2(x^2 - 1063x - 1\,469\,160) = 2(x + 792)(x - 1855) = 0$$

so $x = \underline{1855}$ units. (Of course, this can also be solved using the quadratic formula.)