BLUFFTON UNIVERSITY

MATHEMATICS CONTEST 2015
ANSWER KEY

PART I

Section A

1. 2
2. 10140 minutes
3. $d - m = -4$
4. $x = 0$ or $x = 5$ *(need both, any order)*
5. $2(x + 2)(x - 2)(x^2 + 4)$ *(any order)*
6. 34 units
7. $\sqrt{2}$

Section B

8. $\$62$
9. $a = 19 \quad b = 3 \quad c = 12 \quad d = 6$
10. $(-1, -1)$
11. $\frac{12}{13}$ *or* $12 : 13$
12. $(4, 7)$ and $(5, 4)$ *(need both, any order)*

Section C

13. 12 liters
14. 12 cm
15. $5\frac{1}{2} - \sqrt{3}$ *or* $5.5 - \sqrt{3}$ *or* $\frac{11}{2} - \sqrt{3}$ *or* $\frac{11-2\sqrt{3}}{2}$

PART II

Section A

1. 23
2. 17 years old
3. $7 + 43, 13 + 37, 19 + 31$ *(all, any order)*
4. 135 nickels
5. 2.5
6. 11036
7. A B (C) D E

Section B

8. $2, -2, 4, -4$ *(all four, any order)*
9. $\frac{4}{9}$
10. $k = \frac{13}{4}$ *or* 3.25 *or* $3\frac{1}{4}$
11. 21 square units
12. $x = -\frac{3}{2}$ *or* -1.5

Section C

13. $p = -1$
14. $t = 65536$
15. 21 units

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Simplify: $[2 - 0(1 - 5)]$

Solution: $[2 - 0(1 - 5)] = 2 - 0 = 2$.

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2. One week ago, Duane was at his house on Main Street in Bluffton, Ohio, working on the contest questions. He thought to himself, "I'd better hurry—next Saturday at this time, students will be taking the test!" How many minutes elapsed between 10:00am (Saturday, October 31, 2015) and 10:00am (Saturday, November 7, 2015)?

Solution: Because of "falling back" due to Daylight Saving Time on November 1, there were $7 \cdot 24 + 1 = 169$ hours in that week, which is 10140 minutes.

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3. Consider the numbers $23, 24, 43, 30$ and 25. Let d equal the median and let m equal the mean. Find the value of $d - m$.

Solution: $d = 25$ and $m = 29$, so $d - m = -4$.

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4. Solve for all real values of x: $(x - 2)(x - 3) = 6$

Solution: Expanding the left side of the equation gives $x^2 - 5x + 6 = 6$, or $x^2 - 5x = 0$. Therefore, $x = 0$ or $x = 5$.

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5. Factor completely over the rational numbers: $2x^4 - 32$

Solution: $2x^4 - 32 = 2(x^4 - 16) = 2(x^2 - 4)(x^2 + 4) = 2(x + 2)(x - 2)(x^2 + 4)$.

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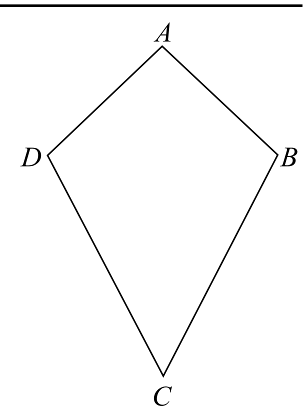
6. (See figure.) Find the numerical value of the perimeter of the kite shown, given these measurements:

$$AB = x + 3 \qquad BC = x + 4$$
$$CD = 2x - 1 \qquad DA = 3x - y$$



Solution: A kite has two (adjacent) pairs of congruent sides, so $AB = DA$ and $BC = CD$. Solving the system

$$x + 3 = 3x - y \quad \text{and} \quad x + 4 = 2x - 1$$

gives $x = 5$ and $y = 7$, meaning that the perimeter is $7x - y + 6 = 34$ units.

7. Express in simplest radical form: $\sqrt{50} - \sqrt{32}$

Solution: $\sqrt{50} - \sqrt{32} = \sqrt{2 \cdot 25} - \sqrt{2 \cdot 16} = 5\sqrt{2} - 4\sqrt{2} = \sqrt{2}$.

Section B. Each correct answer is worth 2 points.

8. The sale ad read, "Buy three baskets at the regular price and get the fourth basket for \$14." I paid \$200 (before tax) for four baskets at the sale. What was the regular price of one basket?

Solution: If x is the regular price of one basket, then $3x + 14 = 200$, so $3x = 186$, and $x = \$62$.

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9. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (20 - 15) \cdot \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, give the values of a, b, c, and d.

Solution: $(20 - 15) \cdot \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 5\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 20-1 & 5-2 \\ 15-3 & 10-4 \end{bmatrix} = \begin{bmatrix} 19 & 3 \\ 12 & 6 \end{bmatrix}$.

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10. Solve for x and y: $3^{3x-1} = 9^{y-1}$ and $8^{2x} = \left(\frac{1}{4}\right)^{y+4}$. Express your answer as an ordered pair (x, y).

Solution: Rewrite the first equation with base 3, and the second with base 2:

$$3^{3x-1} = 9^{y-1} = 3^{2y-2}, \qquad \text{while} \qquad 2^{6x} = 8^{2x} = \left(\frac{1}{4}\right)^{y+4} = 2^{-2y-8}.$$

Equating the exponents yields the system $3x - 1 = 2y - 2$ and $6x = -2y - 8$. Adding the two equations gives $9x - 1 = -10$, so $x = -1$. Substituting in the first equation gives $-4 = 2y - 2$, so $y = -1$.

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11. Find the reciprocal of $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. Express as the ratio of two integers in simplest form.

Solution: $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}$, so the reciprocal is $\frac{12}{13}$.

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12. Define the expression $a \sharp b$ to mean $a^2 + 3b$. There are four pairs of natural numbers (a, b) such that $a \sharp b = 37$. Two such pairs are $(1, 12)$ and $(2, 11)$. Find the other two pairs.

Solution: $a^2 + 3b = 37$ is equivalent to $3b = 37 - a^2$, so we need perfect squares which differ from 37 by a (positive integer) multiple of 3. This happens when $a = 1$ ($3b = 36$), $a = 2$ ($3b = 33$), $a = 4$ ($3b = 21$), and $a = 5$ ($3b = 12$). Therefore, the other pairs are $(4, 7)$ and $(5, 4)$.

Section C. Each correct answer is worth 3 points.

13. How many liters of a 5% acid solution must be added to 6 liters of a 14% acid solution in order to obtain an 8% acid solution?

Solution: x liters of 5% solution contains $0.05x$ liters of acid; similarly, 6 liters of 14% solution contains $(0.06)(14) = 0.84$ liters of acid. If we mix these together, we have $(x + 6)$ liters of solution which contains $0.05x + 0.84$ liters of acid, so we need to solve

$$\frac{0.05x + 0.84}{x + 6} = 0.08.$$

This is equivalent to $0.05x + 0.84 = 0.08x + 0.48$, so $0.03x = 0.36$, and therefore $x = 12$ liters.

14. The scale factor of two spheres is $3:4$. The sum of their volumes is 3276π cm^3. Find the radius, in cm, of the larger sphere.

Solution: If r is the radius of the larger sphere, then the smaller radius is $s = \frac{3}{4}r$, and the sum of the volumes is $\frac{4}{3}\pi r^3 + \frac{4}{3}\pi s^3 = \frac{4}{3}\pi\left(1+\frac{27}{64}\right)r^3 = \frac{91}{48}\pi r^3$. Setting this equal to 3276π leads to $r^3 = 1728$, so $r = 12$ cm.

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15. Express in simplest radical form: $\det\begin{bmatrix}1 & \sqrt{3}\\ 2 & 4\end{bmatrix} + \sum_{k=-1}^{2}\cos(30k^\circ)$

Solution: Evaluating the determinant gives $\det\begin{bmatrix}1 & \sqrt{3}\\ 2 & 4\end{bmatrix} = (1)(4)-(2)(\sqrt{3}) = 4-2\sqrt{3}$, while the sum equals

$$\cos(-30^\circ)+\cos(0^\circ)+\cos(30^\circ)+\cos(60^\circ) = \frac{\sqrt{3}}{2}+1+\frac{\sqrt{3}}{2}+\frac{1}{2} = 1\frac{1}{2}+\sqrt{3}$$

So the total is $5\frac{1}{2}-\sqrt{3}$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Simplify: $\frac{2015}{5} - \frac{1520}{4}$

Solution: $\frac{2015}{5} - \frac{1520}{4} = 403 - 380 = 23$.

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2. Abby, Becka, Carli, and Dani are 15, 16, 17, and 18 years old, but not necessarily in that order. Carli is older than Dani and younger than Abby. Becka is younger than Carli and older than Dani. How old is Carli?

Solution: If Carli is older than Dani and younger than Abby, the only possibilities are

__-D-C-A   D-__-C-A   D-C-__-A   D-C-A-__

Because Becka must be between Dani and Carli, we know that the correct order must be D-B-C-A.

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3. $3+47$ is one of four ways that 50 can be expressed as the sum of two primes. Find the other three.

Solution: Considering the other primes, we find $7+43$, $13+37$, and $19+31$.

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4. Penny has \$20.15 in dimes and nickels. The number of dimes is one fewer than the number of nickels. How many nickels does she have?

Solution: If x is the number of nickels, then $5x+10(x-1) = 2015$, which simplifies to $15x = 2025$, meaning that $x = 135$ nickels.

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5. If $20^x = 2015$, find x rounded off to the nearest tenth.

Solution 1: $x = \log_{20} 2015 = \frac{\ln 2015}{\ln 20} \doteq 2.5397\ldots$.

Solution 2: Use trial and error to approximate the exponent to one decimal place.

6. Find the least common multiple of 124 and 356.

Solution 1: Expressed in prime factors, $124 = 2^2 \cdot 31$ and $356 = 2^2 \cdot 89$. The least common multiple (LCM) must have all of those factors, so it is $2^2 \cdot 31 \cdot 89 = 11036$.

Solution 2: Alternatively, note that the greatest common factor (GCF) is 4, and use the general formula $ab = \mathrm{GCF}(a, b) \cdot \mathrm{LCM}(a, b)$, so $\mathrm{LCM}(124, 356) = \frac{124 \cdot 356}{4} = 11036$.

Solution 3: Many calculators have a built-in function to find the LCM; for example, on the TI-84, it is item 8 on the MATH:NUM menu.

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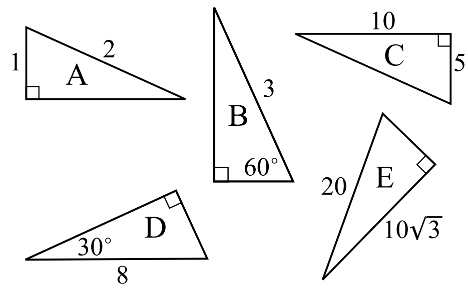
7. (See figure.) On your answer sheet, circle the capital letter of the triangle which is not similar to the other four.



Solution: Obviously, B and D are both 30-60-90 right triangles. In such a triangle, the three side lengths are in the ratio $1 : \sqrt{3} : 2$, which is consistent with A and E; triangle C is the non-similar triangle.

Section B. Each correct answer is worth 2 points.

8. Find the four integer values of x for which $|x^2 - 9|$ is a prime number.

Solution: Note that $|x^2 - 9| = |(x - 3)(x + 3)| = |x - 3||x + 3|$. This can only be a prime number if one of the factors equals 1 (and the other factor is prime); that is, either $|x - 3| = 1$ or $|x + 3| = 1$, which is equivalent to $x - 3 = \pm 1$ or $x + 3 = \pm 1$. Therefore, $x = \pm 2$ or $x = \pm 4$.

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9. Ten marbles numbered 1 to 10 are placed in a jar. Jack reached in and randomly removed one of the marbles and did not replace it. Then Jill reached into the jar and randomly removed another marble. What is the probability that the sum of the two selected marbles is an even number? Express your answer as the ratio of two integers in simplest form.

Solution 1: There are $\binom{10}{2} = 45$ equally likely pairs of numbers, of which $\binom{5}{2} = 10$ are "odd-odd" and another 10 are "even-even." Those 20 pairs have an even sum; the other 25 have an odd sum. So the probability is $\frac{20}{45} = \frac{4}{9}$.

Solution 2: Let E_1 = "the first marble is even," O_1 = "the first marble is odd," with E_2 and O_2 defined similarly for the second marble. Then $P(\text{sum is even}) = P(E_1 \cap E_2) + P(O_1 \cap O_2) = P(E_1)P(E_2|E_1) + P(O_1)P(O_2|O_1) = \frac{1}{2} \cdot \frac{4}{9} + \frac{1}{2} \cdot \frac{4}{9} = \frac{4}{9}$.

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10. The point $(k, 2)$ lies on a line that is perpendicular to the line $3x + 4y = 24$ at the point $(4, 3)$. Find k.

Solution: A perpendicular line would have an equation of the form $4x - 3y = C$. Since it contains the point $(4, 3)$, C must be equal to $4(4) - 3(3) = 7$. Solving $4k - 3(2) = 7$ gives $k = \frac{13}{4}$.

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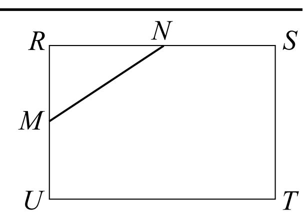
11. (See figure.) The area of rectangle $RSTU$ is 24. M is the midpoint of $\overline{RU}$, and N is the midpoint of $\overline{RS}$. Find the area of pentagon $MNSTU$.



Solution: The area of $\triangle RMN$ is 3 (one-eighth of the area of the rectangle), so the area of pentagon $MNSTU$ is 21 units.

12. A geometric sequence includes these three consecutive terms: x, $x+3$, and $3x+3$. One possible value of x is 3. Find the other possible value of x.

Solution: If these are terms in a geometric sequence, then $\frac{x+3}{x} = \frac{3x+3}{x+3}$; that is, $(x+3)^2 = 3x(x+1)$. Therefore, $0 = 2x^2 - 3x - 9 = (x-3)(2x+3)$; the solutions are $x = 3$ and $x = -\frac{3}{2}$. (Note that the two sequences are $\ldots, 3, 6, 12, \ldots$, and $\ldots, -1.5, 1.5, -1.5, \ldots$.)

Section C. Each correct answer is worth 3 points.

13. One minus the reciprocal of $(1-p)$ equals the reciprocal of $(1-p)$. Find p.

Solution: The equation $1 - \frac{1}{1-p} = \frac{1}{1-p}$ means that $\frac{1}{1-p} = \frac{1}{2}$, so $1 - p = 2$, and therefore $p = -1$.

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14. Find the value of t so that $\log_{16}(\log_2(\log_2 t)) = \frac{1}{2}$.

Solution: Step-by-step, we switch from logarithmic form to exponential form: If $\log_{16} A = \frac{1}{2}$, then $A = 16^{1/2} = 4$. If $\log_2 B = 4$, then $B = 2^4 = 16$. If $\log_2 t = 16$, then $t = 2^{16} = 65536$.

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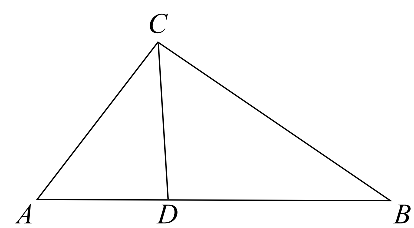
15. (See figure.) Given $\triangle ABC$, with $BC = 12$, $DB = 9$, $CD = 6$, and $m\angle DAC = m\angle BCD$. determine the numerical value of the perimeter of $\triangle ACD$.



Solution 1: $\triangle BCD \sim \triangle BAC$, with ratio of similarity $\frac{BD}{BC} = \frac{9}{12} = \frac{3}{4}$. Therefore $AC = \frac{4}{3} \cdot CD = 8$, $AB = \frac{4}{3} \cdot BC = 16$, and $AD = 7$, and the perimeter of $\triangle ACD$ is $8 + 6 + 7 = 21$ units.

Solution 2: Considering $\triangle BCD \sim \triangle BAC$ as before, we have $\frac{CD}{BC} = \frac{AC}{AB}$, or equivalently, $\frac{6}{12} = \frac{AC}{AD+9}$. This leads to $AC = 8$ and $AD = 7$.