BLUFFTON
EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2014
ANSWER KEY

PART I

Section A

1. -36
2. Between 6 and 7
3. 49 points
4. 25%
5. 99°
6. 9 solutions
7. 50%

Section B

8. $\dfrac{m(m+4)}{m+2}$ *or* $\dfrac{m^2+4m}{m+2}$
9. $(10, -86)$
10. 112 units
11. $y = 8$
12. $b = -2, c = 9$

Section C

13. $\frac{275}{12}$
14. 75π cubic units
15. $x = 13$

PART II

Section A

1. A [✓ B] C D E
2. 310 miles
3. central angle
4. 4 and 15 *either order*
5. $x = 11$ inches
6. $LN = 48$
7. 0

Section B

8. $27, 28, 30$ *(all three, any order)*
9. A B C [✓ D] E
10. $m : n = 861 : 17$
11. 2014_5
12. $34x^2 - 6x + 114$ *or* $2(17x^2 - 3x + 57)$

Section C

13. $137, 173$ *(both, any order)*
14. $g(f(x)) = 6$
15. 10

BLUFFTON

EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2014
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Evaluate: $(20 - 14)(14 - 20)$.

Solution: $(20 - 14)(14 - 20) = (-6)(6) = \underline{-36}$.

2. $\sqrt[3]{300}$ is between what two consecutive integers?

Solution: $6^3 = 216$ and $7^3 = 343$, so the cube root of 300 is between 6 and 7.

3. Player P scored 3 points, Player Q scored three times as many as Player P, Player R scored 3 more points than Player P, Player S scored twice as many points as Player Q, and Player T scored 13 points. How many total points were scored by Players P, Q, R, S, and T?

Solution: P scored 3, Q scored 9 (three times as many as P), R scored 6 (three more points than P), S scored 18 (twice as many as Q), and T scored 13, so their total was 49.

4. Last year we had 20. This year we have 15. What is the percent of decrease?

Solution: Percent of decrease is change divided by original amount, which is $5/20 = \underline{25\%}$.

5. The measure of an angle is 54°. Find the average of its supplement and twice its complement.

Solution: The supplement is 126°, the complement is 36°, and the average is $\frac{126+72}{2} = \underline{99^\circ}$.

6. How many positive integer solutions exist for $3(x - 8) \leq 5$?

Solution: $3(x - 8) \leq 5$ is equivalent to $3x \leq 29$, or $x \leq 9\frac{2}{3}$, so there are $\underline{9}$ positive integer solutions.

7. Three integers are chosen at random from $\{1, 2, 3, \ldots, 1000\}$. (Repetition is allowed; that is, a number may be chosen more than once.) What is the probability that their sum is *even*? Express your answer as a *percent*.

Solution 1: Each integer is even or odd with equal probability. Therefore, the sum of a pair of such integers is even or odd with equal probability, and likewise for the sum of three such integers, so the probability is 50%.

Solution 2: The sum is even if all three integers are even, or if exactly two are odd. Therefore, $P(\text{sum is even}) = P(\text{all even}) + P(\text{one even, two odd}) = \left(\frac{1}{2}\right)^3 + \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{1}{2}$ or $\underline{50\%}$.

Solution 3: For each number chosen, if we consider only even vs. odd, the sample space (set of possible outcomes) is { EEE, EEO, EOE, OEE, EOO, OEO, OOE, OOO }. Four of these eight (equally likely) outcomes result in an even total.

Section B. Each correct answer is worth 2 points.

8. Simplify the fraction: $\dfrac{m^3 + 2m^2 - 8m}{m^2 - 4}$

Solution: Factoring the numerator and denominator, then canceling common factors, gives $\dfrac{m(m^2 + 2m - 8)}{(m-2)(m+2)} = \dfrac{m(m+4)(m-2)}{(m-2)(m+2)} = \dfrac{m(m+4)}{m+2}$ or $\dfrac{m^2 + 4m}{m+2}$.

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9. Find the vertex of the parabola $y = x^2 - 20x + 14$. Write as an ordered pair (h, k).

Solution 1: Completing the square gives $y = x^2 - 20x + 100 - 86 = (x - 10)^2 - 86$, so the vertex is $\underline{(10, -86)}$.

Solution 2: For a parabola $y = ax^2 + bx + c$, the x-coordinate of the vertex is $x = -\frac{b}{2a}$, so $h = 10$, and $k = 10^2 - 20 \cdot 10 + 14 = -86$.

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10. A rectangle with whole-number dimensions has an area of 72 square units. Let M be the maximum perimeter for such a rectangle, and m be the minimum perimeter. What is $M - m$?

Solution: Such a rectangle must have dimensions a and b, for which $ab = 72$. By listing factors of 72, we can list all possible dimensions; here they are as ordered pairs (a, b), with $a < b$:

$$(1, 72), \qquad (2, 36), \qquad (3, 24), \qquad (4, 18), \qquad (6, 12), \qquad (8, 9)$$

The ordered pair $(1, 72)$ produces the maximum perimeter ($M = 146$ units), while $(8, 9)$ produces the minimum perimeter ($m = 34$ units), so $M - m = \underline{112 \text{ units}}$.

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11. Find y so that the points $A(0, 0)$, $B(2, 2)$ and $C(-4, y)$ are the vertices of a right triangle with hypotenuse $\overline{AC}$.

Solution: If $\overline{AC}$ is the hypotenuse, then $\angle B$ is the right angle, so $\overline{BC}$ must be perpendicular to $\overline{AB}$ (which lies on a line with slope 1). Therefore, C lies on the line through B with slope -1. This line has equation $y = 4 - x$, so $y = 4 - (-4) = \underline{8}$.

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12. The polynomial equation $x^3 + bx^2 + cx - 18 = 0$ has real coefficients, and the complex number $3i$ is a root. Find the values of b and c.

Solution 1: If a polynomial with real coefficients has $3i$ as a root, then its conjugate $-3i$ must also be a root, and there must also be a real root (say, k). Therefore, the polynomial factors into $(x - 3i)(x + 3i)(x - k) = (x^2 + 9)(x - k)$. This product has constant term $-9k$, so k must equal 2. Multiplying gives $x^3 - 2x^2 + 9x - 18$, so $\underline{b = -2}$ and $\underline{c = 9}$.

Solution 2: With $f(x) = x^3 + bx^2 + cx - 18$, we have

$$f(3i) = -27i - 9b + 3ci - 18 = (-9b - 18) + (3c - 27)i = 0 + 0i.$$

Equating (separately) the real and imaginary parts, we have $-9b - 18 = 0$ and $3c - 27 = 0$. Therefore, $b = -2$ and $c = 9$. *Note:* If we had not been told that b and c were real, we would not have known that (e.g.) $-9b - 18$ is real, and there would be an infinite number of (complex) answers; for example, $b = 1 - i$ and $c = 6 - 9i$.

<u>Section C</u>. *Each correct answer is worth 3 points.*

13. Express as a ratio of two integers in simplest form: $\dfrac{\left(\frac{2}{3}\right)^{-2}+\left(\frac{3}{2}\right)^{-3}}{3^{-2}}$

<u>Solution:</u> $\dfrac{\left(\frac{2}{3}\right)^{-2}+\left(\frac{3}{2}\right)^{-3}}{3^{-2}} = 3^2\left[\left(\frac{3}{2}\right)^2+\left(\frac{2}{3}\right)^3\right] = 9\left(\frac{9}{4}+\frac{8}{27}\right) = \frac{81}{4}+\frac{8}{3} = \frac{243}{12}+\frac{32}{12} = \frac{275}{12}$.

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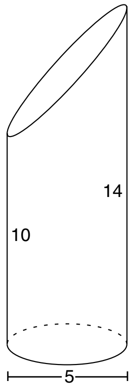
14. (See figure.) Find the exact volume of the cylinder cut on a slant.



<u>Solution 1:</u> The volume of a right circular cylinder (without the slant cut) would be $V = \pi r^2 h$. Just as one computes the area of a trapezoid as the average height times the base (or average base times height), we account for the effect of the slant cut by using the average height of the cylinder for h. To see why this works, imagine slicing off the top of the slant at a height of 12 units up from the base. The cut-off piece could then be flipped upside down and "glued" onto the bottom to make a cylinder with a height of 12 units. Therefore, the area is $V = \pi(2.5)^2(12) = \pi\left(\frac{25}{4}\right)(12) = \underline{75\pi \text{ cubic units}}$.

<u>Solution 2:</u> Alternatively (but equivalently), if we slice off the slanted top part, we have a complete cylinder (radius 2.5, height 10, volume $\pi \cdot 2.5^2 \cdot 10$) and a half-cylinder (radius 2.5, height 4, volume $0.5\pi \cdot 2.5^2 \cdot 4$), so the total volume is 75π.

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15. Solve for x if $x - 5 = \det \begin{vmatrix} \log_4 32 + \log_3 27 + \log_9 3 & \dfrac{(11-3)!}{5!\cdot 6\cdot 7} \\ 2\cos(0^\circ) - \sin(0^\circ) & \displaystyle\sum_{k=-1}^{2} |k| \end{vmatrix}$

<u>Solution:</u> Simplify each term in the matrix:

- $\log_4 32 + \log_3 27 + \log_9 3 = \log_4(4^{5/2}) + \log_3(3^3) + \log_9(9^{1/2}) = 2.5 + 3 + 0.5 = 6$,
- $\frac{(11-3)!}{5!\cdot 6\cdot 7} = \frac{8!}{7!} = 8$,
- $2\cos(0^\circ) - \sin(0^\circ) = 2\cdot 1 - 0 = 2$, and
- $\displaystyle\sum_{k=-1}^{2} |k| = |-1| + |0| + |1| + |2| = 4$.

Then $x = 5 + \det\begin{vmatrix} 6 & 8 \\ 2 & 4 \end{vmatrix} = 5 + 8 = \underline{13}$.

PART II: 30 Minutes; CALCULATORS NEEDED

<u>Section A</u>. *Each correct answer is worth 1 point.*

1. $(20+14)(15+16) = (14+20)(15+16)$ is an example of the ________ property of addition. Circle the letter of the term that correctly fills in this blank.

A) associative B) commutative C) distributive D) identity E) inverse

<u>Solution:</u> The left side has "20+14" while the right side has "14+20," which uses the <u>commutative</u> property of addition.

2. Driving at a constant rate of speed, Danica drove 180 miles in 3 hours. She then increased her rate 5 mph and drove 2 more hours. What was the total distance that she drove?

Solution: If Danica drove 180 miles in three hours, she was traveling at 60 mph. Driving two more hours at 65 mph would take her another 130 miles, for a total of 310 miles.

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3. What is the name of an angle whose vertex is the center of a circle, and whose sides are radii of that circle?

Solution: If the vertex of an angle falls on the center of a circle (in the same plane), it is a central angle of that circle.

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4. The set of seven natural numbers { 6, ___, 4, 9, ___, 8, 3 } has mean 7 and mode 4. What are the two missing numbers? (Order does not matter.)

Solution: The five given numbers add up to 30, so if the mean is 7, the missing numbers must total 19. If the mode is 4, one of the missing numbers is 4, so the other is 15.

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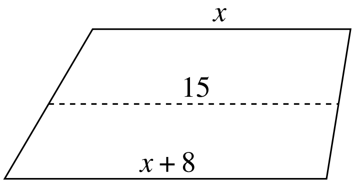
5. A trapezoid has height 10 inches, midsegment (median) of length 15 inches, and bases of x inches and $(x + 8)$ inches. Find x.

Solution: The midsegment (median) of a trapezoid connects the midpoints of the trapezoid's *legs* (the non-parallel sides). It is parallel to the bases, and more importantly, its length is the average of the base lengths. Therefore, $15 = \frac{2x+8}{2}$, so $x = 11$ inches. (Note: The height of the trapezoid is not needed .)



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6. Given $\triangle RST \sim \triangle LNE$ with $RT = 10, TS = 9, RS = 8, EN = 6x - 12$, and $LE = 5x + 5$, find the numerical value of LN.

Solution: From the proportion $\frac{TS}{EN} = \frac{RT}{LE}$—or equivalently, $TS \cdot LE = RT \cdot EN$—we have $9(5x + 5) = 10(6x - 12)$. Multiplying out and combining like terms gives $15x = 165$, so $x = 11$. Therefore, $LE = 60$ (and $EN = 54$), meaning that the sides of $\triangle LNE$ are 6 times as long as the corresponding sides of $\triangle RST$, so $LN = 48$.

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7. Find the slope of any line that is perpendicular to the line $x - 20 = 14$.

Solution: $x - 20 = 14$ (or $x = 34$) is a vertical line, so lines that are perpendicular are horizontal, and thus have slope 0.

Section B. Each correct answer is worth 2 points.

8. List all the integers from 25 to 30, inclusive, that cannot be expressed as the sum of two squares. (For example, $5 = 2^2 + 1^2$ can be expressed as the sum of two squares, but 6 cannot be expressed this way.)

Solution: We only need to consider sums of $1^2, 2^2, 3^2, 4^2$, and 5^2: $25 = 3^2 + 4^2$ (or $0^2 + 5^2$), $26 = 1^2 + 5^2$, and $29 = 2^2 + 5^2$. The other three numbers—27, 78, 30—cannot be written as the sum of two squares.

9. On the answer sheet, circle the letter of the one expression which is not a factor of $x^6 - y^6$.

A) $x^2 - y^2$ B) $x^3 + y^3$ C) $x^4 + x^2y^2 + y^4$
D) $x^2 + 2xy + y^2$ E) $x^3 - 2x^2y + 2xy^2 - y^3$

Solution 1: The answer is D. One way to see this is to factor $x^6 - y^6$ completely into non-reducible expressions (i.e., those that cannot be factored further), which gives $x^6 - y^6 = (x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$. Noting that D $= (x + y)^2$, there is no way to obtain D by multiplying these factors. As for the other options, if we call these factors F_1, F_2, F_3 and F_4, respectively, then A $= F_1 \cdot F_2$, B $= F_2 \cdot F_4$, C $= F_3 \cdot F_4$, and E $= F_1 \cdot F_4$.

Solution 2: You can perform long division to determine if an expression is a factor. For example, $(x^6 - y^6) \div (x^2 - y^2) = x^4 + x^2y^2 + y^4$, which eliminates both A and C as answers.

10. If $2^3(20m + 14n) - 3^2(14m - 20n) = 2014n$, find the ratio of m to n, expressed as a ratio of two integers in simplest form.

Solution: Simplify and combine like terms: $8(20m + 14n) - 9(14m - 20n) = 160m + 112n - 126m + 180n = 2014n$, so $34m = 1722n$. Therefore, $\frac{m}{n} = \frac{1722}{34} = \frac{861}{17}$. Therefore, $m : n = 861 : 17$.

11. Convert from base 8 to base 5: $403_8 =$ ______$_5$.

Solution: $403_8 = 4 \cdot 8^2 + 0 \cdot 8^1 + 3 \cdot 8^0 = 259$ (base 10). The powers of 5 are $1, 5, 5^2 = 25$, $5^3 = 125$, $5^4 = 625$, so we have 2 groups of 125, plus 1 group of 5, and 4 left over; that is, $259_{10} = 2 \cdot 5^3 + 0 \cdot 5^2 + 1 \cdot 5^1 + 4 \cdot 5^0 = 2014_5$.

12. Subtract the opposite of $(14x^2 - 20x + 7^2)$ from $(20x^2 + 4^3 + 14x + 5^0)$, and simplify.

Solution: "Subtract the opposite" means "add," so we have

$$(14x^2 - 20x + 7^2) + (20x^2 + 4^3 + 14x + 5^0) = 14x^2 - 20x + 49 + 20x^2 + 64 + 14x + 1$$

which equals $34x^2 - 6x + 114$.

Section C. Each correct answer is worth 3 points.

13. List all prime numbers between 100 and 199 such that the tens digit is a prime number, the units digit is a prime number, and the tens and units digit taken together are a two-digit prime number.

Solution: The tens digit must come from the set of one-digit primes $\{2, 3, 5, 7\}$. The units digit must be either 3 or 7 (since a two- or three-digit number ending in 2 or 5 is not prime). This leaves only eight possible candidates:

123	no – divisible by 3
127	no – 27 is not prime
133	no – 33 is not prime
137	YES
153	no – divisible by 3
157	no – 57 is not prime
173	YES
177	no – 77 is not prime

So the answers are 137 and 173.

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14. Given: $f(x) = x^2 - 20x + 11$ and $g(x) = \sqrt{x+4}$. If $g(x) = 5$, find $g(f(x))$.

Solution: If $\sqrt{x+4} = 5$, then $x = 21$. Therefore, $f(x) = f(21) = 32$, and $g(f(x)) = g(32) = \sqrt{36} = \underline{6}$.

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15. Let $5, m, 20, \ldots$ be an arithmetic sequence. Let $5, n, 20, \ldots$ (with $n \geq 0$) be a geometric sequence. Let $5, p, q, 20, \ldots$ be a Fibonacci-style sequence. Write the value of $m - n + p$.

Solution 1: In an arithmetic sequence, we add a constant to get the next term, so $5 + c = m$ and $m + c = 20$. Therefore, $c = m - 5$, so $2m - 5 = 20$, so $m = 12.5$.

In a geometric sequence, we multiply by a constant to get the next term, so $5r = n$ and $nr = 20$. Therefore, $r = n/5$, so $n^2/5 = 20$, so $n = 10$.

In a Fibonacci-style sequence, we add two consecutive items to get the next item, so $5+p = q$ and $p + q = 20$. Therefore, $p + (5 + p) = 20$, so $p = 7.5$.

Finally, $m - n + p = 12.5 - 10 + 7.5 = \underline{10}$.

Solution 2: Alternatively, note that a term in an arithmetic sequence is always the average (arithmetic mean) of its neighbors. Similarly, a term in a geometric sequence is always the *geometric* mean of its neighbors, and in a Fibonacci-style sequence a, b, c, d, the two middle terms are related to their surrounding terms by $b = (d - a)/2$ and $c = (d + a)/2$. Therefore, $m = \frac{5+20}{2} = 12.5$, $n = \sqrt{5 \cdot 20} = 10$, and $p = \frac{20-5}{2} = 7.5$.