BLUFFTON
EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2013
ANSWER KEY

PART I

Section A

1. 16
2. x could be $2, 3, 4, 5, 6$ (any order)
3. $\frac{7}{16}$
4. $6 + 4x$ *or* $4x + 6$
5. 13 *or* $\sqrt{169}$ units
6. 100
7. $x = -4$

Section B

8. 75%
9. Wednesday
10. 21
11. 43 units
12. $a = -18$

Section C

13. $x = \frac{3}{4}$ *or* 0.75
14. $-2015 - 2i$
15. $b = -2$ $c = 3$ $d = -2$

PART II

Section A

1. 140°
2. $a = 7$
3. CCCLXIV
4. obtuse
5. $\pi - 3$ *or* $-(3 - \pi)$
6. 6
7. 26.8

Section B

8. 3
9. 1
10. $x = -1$
11. $m\angle P = 30^\circ$ $m\angle BFC = 50^\circ$
12. 100 guppies

Section C

13. 18.2 units
14. (2, 1)
15. 188 feet

MATHEMATICS CONTEST 2013
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Find the sum of all the positive integral factors of 12, excluding the number 12 itself.

Solution: The (positive integral) factors of 12 are $\{1, 2, 3, 4, 6, 12\}$; the sum of these factors (excluding 12) is $\underline{16}$.

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2. The sides of a triangle are all integers. If the 3 sides are $3, 4$, and x, list all possible values of x.

Solution: If a, b, and c are the side lengths of a triangle, then they must satisfy the triangle inequalities: $a + b > c$, $a + c > b$, and $b + c > a$. With $a = 3$, $b = 4$, and $c = x$, that means $7 > x$, $3 + x > 4$, and $4 + x > 3$. The latter two inequalities mean that $x > 1$ and $x > -1$, so we can ignore the third, and conclude that x is an integer satisfying $1 < x < 7$; that is, $\underline{\{2, 3, 4, 5, 6\}}$.

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3. One stamp was randomly selected from an 8×8 sheet of 64 stamps. Find the probability that the stamp was one of this sheet's border stamps. Express as a fraction in simplest form.

Solution: There are 28 stamps on the border of the sheet, so the probability of selecting one of them is $\frac{28}{64} = \frac{7}{16}$.

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4. Express in simplest form: $0x^1 + 2^0 + 4x^0 + (4x)^0 + 4x$

Solution: $0x^1 + 2^0 + 4x^0 + (4x)^0 + 4x = 0 + 1 + 4 + 1 + 4x = \underline{6 + 4x}$.

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5. Find the length of a diagonal of a rectangle whose perimeter is 34 and whose width is 5.

Solution: We have perimeter $P = 34 = 2L + 2W$, with width $W = 5$, so the length of the rectangle is $L = 12$, and the diagonal is $\sqrt{L^2 + W^2} = \sqrt{5^2 + 12^2} = \underline{\sqrt{169}} = \underline{13}$ units.

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6. Express in simplest form: $\sqrt{2} \cdot \sqrt{4} \cdot \sqrt{5} \cdot \sqrt{10} \cdot \sqrt{25}$

Solution: $\sqrt{2} \cdot \sqrt{4} \cdot \sqrt{5} \cdot \sqrt{10} \cdot \sqrt{25} = 2 \cdot 5 \cdot \sqrt{2 \cdot 5 \cdot 10} = 10 \cdot \sqrt{100} = \underline{100}$.

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7. Solve for x: $3(x + 2) = x - 6 + 4$

Solution: Expand the left side, and simplify the right: $3x + 6 = x - 2$. Now collect like terms: $2x = -8$. Therefore, $\underline{x = -4}$.

Section B. Each correct answer is worth 2 points.

8. Two positive integers are chosen at random. What is the probability that their product is *even*? Express your answer as a *percent*.

Solution: The product of two integers will be even if either one (or both) is even—or to put it another way, the product is odd only if both integers are odd. The chance both are odd is $(50\%)(50\%) = 25\%$, and therefore the probability that the product is even is 75%.

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9. The International Department of Climate Control decreed that there would no longer be any weather on even-numbered calendar dates. During one of the months following that decree, three Mondays had no weather. For that month, on what day of the week did the 18^{th} occur?

Solution: If there were three even-numbered Mondays in that month, there must have been five Mondays in total, and the even Mondays fell on the 2^{nd}, 16^{th}, and 30^{th}. Therefore, the 18^{th} was a Wednesday.

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10. Consider all of the two-digit numbers that can be made from the digits 2, 0, 1, and 3. (Do not use any digit twice, like '22,' and do not begin with a zero, like '03.') Which of these numbers is closest to the mean (average) of these numbers?

Solution: There are nine such numbers; in order, we have $10, 12, 13, 20, 21, 23, 30, 31, 32$. They add up to 192, so the mean is $21\frac{1}{3}$, which is closest to 21.

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11. Trapezoid *TRAP* has base $TR = y + 6$ and base $PA = 5y + 8$. If the length of the median (midsegment) is 28, find the length of the longer base.

Solution: The length of a trapezoid's median (midsegment) is the average of the lengths of its bases, so $28 = \frac{TR+PA}{2} = \frac{6y+14}{2} = 3y + 7$. Therefore, $y = 7$, and the longer base is $PA = 5 \cdot 7 + 8 = 43$.

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12. The line $y = 3x + 6$ is parallel to the line $ax + 6y = 18$. Find the value of a.

Solution: The line $ax + 6y = 18$ has slope $m = -\frac{1}{6}a$, so $-\frac{1}{6}a = 3$, and therefore $a = -18$.

Section C. Each correct answer is worth 3 points.

13. $\log_2 64 + \log_3 27 + \log_6 1 - \log_{12} 12 = 16^x$. Find x.

Solution: $\log_2 64 + \log_3 27 + \log_6 1 - \log_{12} 12 = 6 + 3 + 0 - 1 = 8$, so we have $8 = 16^x$. Rewriting as powers of 2 gives $2^3 = 2^{4x}$, so $3 = 4x$, and therefore $x = \frac{3}{4}$.

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14. If $i = \sqrt{-1}$, simplify $2013i + 2014i^2 + 2015i^3 + i^{2013} + i^{2014} + i^{2015}$.

Solution: Powers of i repeat in cycles of 4: $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$, and so on. Because 2013 is one more than a multiple of 4 (that is, $2013 = 4 \cdot 503 + 1$), $i^{2013} = i^1$, and likewise, $i^{2014} = i^2$ and $i^{2015} = i^3$. So, the given expression simplifies to

$$2013i + 2014(-1) + 2015(-i) + i + (-1) + (-i) = 2014i - 2015 - 2016i = -2015 - 2i$$

15. In the system of equations on the right, if $a = 1$, find the values of b, c, and d.

$$\begin{cases} 4a + 3b + 2c + \ \ d = 2 \\ 3a + 2b + \ \ c + 4d = -6 \\ 2a + \ \ b + 4c + 3d = 6 \end{cases}$$

Solution: With $a = 1$, the system reduces to

$$\begin{aligned} 3b + 2c + \ \ d &= -2 \\ 2b + \ \ c + 4d &= -9 \\ b + 4c + 3d &= 4 \end{aligned}$$

which can be solved by any of the standard methods (eliminating variables, Cramer's rule, etc.) For example, from the third equation, we have $b = 4 - 4c - 3d$; substituting into the first and second equations leaves

$$-10c - 8d = -14 \qquad \text{and} \qquad -7c - 2d = -17.$$

Divide the first equation by 2, and multiply the second equation by -2, then add the results to get $9c = 27$, so $\underline{c = 3}$. Substituting in $-7c - 2d = -17$ then gives $= 2d = 4$, so $\underline{d = -2}$, and therefore $\underline{b} = 4 - 4c - 3d = 4 - 4(3) - 3(-2) \underline{= -2}$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Find the supplement of the complement of an angle whose measure is 50°.

Solution: The complement of a 50° angle has measure 40°. Its supplement has measure 140°.

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2. Given that the 6-digit number $7a2013$ is divisible by 11, what is the value of a?

Solution 1: The standard test for divisibility by 11 is: A number is divisible by 11 if and only if, when you alternately add and subtract the digits (in the order they appear), the result is divisible by 11. In this case, that means that $7 - a + 2 - 0 + 1 - 3 = 7 - a$ must be divisible by 11. The only single-digit value of a that makes this true is $\underline{a = 7}$.

Solution 2: Note that $2013 = 83 \cdot 11$, so $7a2013$ is divisible by 11 if and only if $7a0000$ is divisible by 11, which (again) implies that $\underline{a = 7}$.

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3. Roman math teacher Mathematicus assigned this problem: Multiply (XIV)·(XXVI). Do this multiplication, and express your answer using Roman numerals.

Solution: (XIV)·(XXVI) $= 14 \cdot 26 = 364 =$ CCCLXIV.

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4. Give the name for an angle whose measure is between 90° and 180°.

Solution: Such angles are called obtuse.

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5. Give the exact value of $|3 - \pi|$.

Solution: If $x < 0$, then $|x| = -x$. Because $3 - \pi$ is negative, $|3 - \pi| = -(3 - \pi) = \underline{\pi - 3}$.

6. In the stem-and-leaf plot (also called a stemplot) on the right, "0 | 8" represents the number 8. Find the difference between the range and the median.

0	8 9 9
1	2 2 5 6
2	3 7 8 9

Solution: The stem-and-leaf plot represents 11 observations. The range is the difference between the maximum and the minimum: $29 - 8 = 21$. The median is the middle (6^{th}) number, which is 15. So, range minus median $= 21 - 15 = \underline{6}$.

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7. Find the value of $\sqrt{6!}$ to the nearest tenth.

Solution: $\sqrt{6!} = \sqrt{720} = 26.8328\ldots$; rounded to the nearest tenth, that is $\underline{26.8}$.

Section B. Each correct answer is worth 2 points.

8. Given a hexagon, let S be the number of sides, and let D be the number of diagonals. Find $20S - 13D$.

Solution: A hexagon has $S = 6$ sides and $D = 9$ diagonals*, so $20S - 13D = \underline{3}$.

*If the vertices of the hexagon are consecutively labeled A, B, C, D, E, and F, then the diagonals are AC, AD, AE, BD, BE, BF, CE, CF, and DF.

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9. What is the remainder when 13^{20} is divided by 5?

Solution 1: If $a \div n$ and $b \div n$ have the same remainder (that is, if $a - b$ is divisible by n), we write "$a \equiv b \pmod{n}$," which is read as "a is congruent to b mod n." A useful fact about divisibilty and congruence is that, if $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$.

Because $13 \equiv 3 \pmod{5}$, it must also be true that $13^{20} \equiv 3^{20} \pmod{5}$. Furthermore, $3^{20} \equiv 9^{10} \equiv 4^{10} \equiv 16^5 \pmod{5}$, which is congruent to $\underline{1}$.

Solution 2: (This uses the same reasoning as solution 1, but without the "congruent mod 5" language.) The remainder when dividing by 5 depends only on the last digit of 13^{20}. So we observe that

$$\text{last digit of } 13^{20} = \text{last digit of } 3^{20} = \text{last digit of } 9^{10} = \text{last digit of } 81^5$$

which is $\underline{1}$.

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10. If $M \# N = 2M + (5 - N)$, find x such that $3 \# x = 12$.

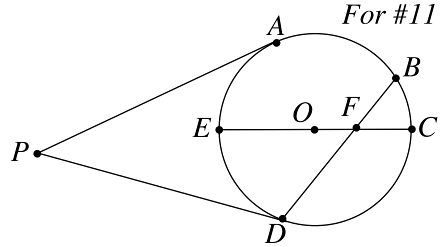
Solution: According to the given definition, $3 \# x = 2 \cdot 3 + (5 - x) = 11 - x$. If this is to equal 12, x must be $\underline{-1}$.

11. In the figure shown, $\overline{PA}$ and $\overline{PD}$ are tangent to $\odot O$ at A and D, respectively. $m\widehat{AB} = 90°$, $m\widehat{BC} = 20°$, and $m\widehat{CD} = 100°$. Find $m\angle P$ and $m\angle BFC$.

Solution: Using the given arc measures, we find that $m\widehat{ACD} = 210°$, $m\widehat{AD} = 150°$, and $m\widehat{DE} = 80°$, so

$$m\angle P = \frac{1}{2}\left(m\widehat{ACD} - m\widehat{AD}\right) = \underline{30°}, \text{ and } m\angle BFC = \frac{1}{2}\left(m\widehat{BC} + m\widehat{DE}\right) = \underline{50°}.$$

For #11



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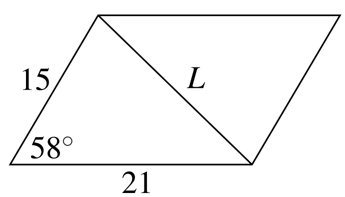
12. There are 200 fish in an aquarium. 99% of them are guppies. How many guppies must be removed to reduce the percentage of guppies to 98%?

Solution: The aquarium starts with 198 guppies and 2 non-guppies. To have 98% guppies (and 2% non-guppies), we need to remove <u>100 guppies</u> (leaving 100 fish total).

<u>Section C</u>. Each correct answer is worth 3 points.

13. Two adjacent sides of a parallelogram have lengths of 15 and 21. The smaller angle formed by the sides has a measure of 58°. Find the length of the shorter diagonal; express your answer to the nearest tenth.

Solution: The shorter diagonal forms a triangle with side lengths 15 and 21, and included angle 58° (see figure below), so using the law of cosines, we have $L^2 = 15^2 + 21^2 - 2 \cdot 15 \cdot 21 \cdot \cos 58° = 332.1508\ldots$. Therefore, $L = 18.2250\ldots$, which rounds to <u>18.2 units</u>.



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14. Given the ellipse $25x^2 + 9y^2 - 100x + 54y - 44 = 0$, find the coordinates (x, y) of the focus point lying in the first quadrant.

Solution: Completing the squares leads to $25(x^2 - 4x + 4) - 100 + 9(y^2 + 6y + 9) - 81 - 44 = 0$, or $25(x-2)^2 + 9(y+3)^2 = 225$. In standard form, this is

$$\frac{(x-2)^2}{3^2} + \frac{(y+3)^2}{5^2} = 1$$

so the ellipse has semi-minor axis length $a = 3$ and semi-major axis length $b = 5$. The foci are located at a distance of $\pm c$ above and below the center $(2, -3)$, where $c = \sqrt{b^2 - a^2} = 4$, so the focus in the first quadrant is at <u>(2, 1)</u>.

15. Several rectangles with integral sides each have an area of 2013 square feet. Of all such rectangles, find the one that has the smallest perimeter. How many feet are in that perimeter?

Solution: The prime factorization of 2013 is $3 \cdot 11 \cdot 61$, so there are only four rectangles with integral sides and an area of 2013 ft^2; their dimensions are 1×2013, 3×671, 11×183, and 33×61. The last of these has the smallest perimeter: $2 \cdot 33 + 2 \cdot 61 =$ 188 feet.