BLUFFTON

EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2012
ANSWER KEY

PART I

Section A

1. 1
2. 20 miles
3. diagonal
4. 6
5. (a) N (b) S
6. $y = 0.8x + 10.4$
7. 1.4×10^6

Section B

8. -4
9. 2
10. $x = 2$ *or* 2 *or* $\{2\}$
11. 15
12. $x = 20$ $y = 20$

Section C

13. $x = 3$ *or* 3 *or* $\{3\}$
14. $\frac{4}{7}$
15. $k = -8$

PART II

Section A

1. 98573
2. 6
3. 108°
4. 90°
5. 16 feet
6. $m = 8$
7. $\frac{113}{183}$

Section B

8. 16
9. 174
10. $p = 28$
11. $-14\frac{2}{3}$ *or* $-14.\overline{6}$ *or* $-\frac{44}{3}$
12. $6\sqrt{2}$ *or* $\sqrt{72}$ *or* 8.4853 units

Section C

13. $c = 49$
14. 144π square cm
15. $(0, -6), (0, 2), (6, -6), (6, 2)$ *(any order)*

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Simplify: $2 \cdot 0 + 1^2$

Solution: $2 \cdot 0 + 1^2 = 0 + 1 = 1$

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2. Einstein can travel 5 mi in 3 hours. At the same rate, how many miles can he travel in 12 hours?

Solution: Set up a proportion: $\frac{5 \text{ miles}}{3 \text{ hours}} = \frac{x}{12 \text{ hours}}$. Solving (for example, by cross-multiplying) gives $x = 20$ miles.

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3. Give the name of this geometric term: A segment joining two nonconsecutive vertices of a plane polygon is called a(n) ________.

Solution: A segment joining two nonconsecutive vertices of a plane polygon is a diagonal.

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4. Express in simplest form: $\dfrac{1}{2} + \dfrac{3}{0.3} - 4.5$.

Solution: $\dfrac{1}{2} + \dfrac{3}{0.3} - 4.5 = 0.5 + 10 - 4.5 = 6$.

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5. Answer each of the following with A (Always), S (Sometimes), or N (Never). (Both answers must be correct to receive credit.)
   (a) Two lines parallel to a third line are A-S-N perpendicular to each other.
   (b) Two lines perpendicular to a third line are A-S-N parallel to each other.

Solution: (a) Two lines parallel to a third line are never perpendicular to each other. (b) This statement is always true in the plane, but fails if the three lines are not coplanar (that is, it fails in three dimensions). There, the first two lines might be skew (rather than parallel), or the three lines might intersect in a single point (like the x, y and z axes). So overall, it is sometimes true.

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6. Find the equation of the line that passes through $(12, 20)$ and $(-13, 0)$. Express in slope-intercept form $y = mx + b$, with m and b given as decimals.

Solution: The slope of this line is $m = \frac{20-0}{12-(-13)} = \frac{20}{25} = 0.8$, so the y-intercept is $b = 20 - 12m = 20 - 9.6 = 10.4$. Therefore, the equation is $y = 0.8x + 10.4$.

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7. Simplify and write the answer in scientific notation: $(3.5 \times 10^4) \times (4 \times 10^1)$.

Solution: $(3.5 \times 10^4) \times (4 \times 10^1) = (3.5 \times 4) \times (10^4 \times 10^1) = 14 \times 10^5 = 1.4 \times 10^6$.

Section B. *Each correct answer is worth 2 points*.

8. Simplify: $-3^2 + 3^0 + 3x^0 + (3x)^0$

Solution: $-3^2 + 3^0 + 3x^0 + (3x)^0 = -9 + 1 + 3 + 1 = -4$

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9. Evaluate: $\sin(30^\circ) + \tan(45^\circ) + \cos(60^\circ) + \sin(90^\circ) + \cos(120^\circ) + \tan(135^\circ) + \sin(150^\circ)$

Solution: The first three trig function values can be found using right triangles (30-60-90 and 45-45-90). The fourth term is $\sin(90^\circ) = 1$. The last three can be found using reference triangles in the second quadrant (where the sine is positive and the cosine and tangent are negative). Therefore: $\sin(30^\circ) + \tan(45^\circ) + \cos(60^\circ) + \sin(90^\circ) + \cos(120^\circ) + \tan(135^\circ) + \sin(150^\circ) = \frac{1}{2} + 1 + \frac{1}{2} + 1 - \frac{1}{2} - 1 + \frac{1}{2} = 2$.

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10. Solve for x: $\log_6(x) + \log_6(x+1) = 1$.

Solution: Using the product rule for logarithms, $\log_6(x) + \log_6(x+1) = \log_6[x(x+1)] = 1$. Exponentiating both sides (with base 6) leads to $x(x+1) = 6$, so $x^2 + x - 6 = 0$, or $(x-2)(x+3) = 0$. Of these two solutions, $x = -3$ is extraneous because it is not valid in the original equation. Therefore, the only solution is $x = 2$.

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11. The sum of the digits of a three-digit number is 5. How many such three-digit numbers are there? (Note: A three-digit number does not begin with zero.)

Solution: Consider each of the possible sets of three digits which sum to 5: $0+0+5$ (500), $0+1+4$ (104, 140, 401, 410), $0+2+3$ (four more possibilities), $1+1+3$ (113, 131, 311), $1+2+2$ (three more possibilities). That gives a total of 15 such numbers.

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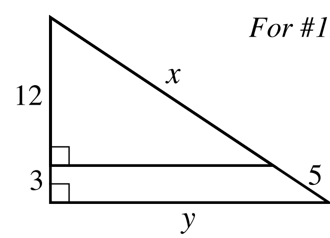
12. In the figure on the right, find x and y.

For #12



Solution: Set up proportions using the similar triangles: $\frac{x}{12} = \frac{x+5}{15}$, so $15x = 12x + 60$, so $x = 20$. Then the larger triangle has legs 15 and y, and hypotenuse 25, so (using the Pythagorean theorem) $y = 20$.

Section C. *Each correct answer is worth 3 points*.

13. Solve for all real values of x: $\sqrt{x-3} - \sqrt{x+1} = -2$

Solution: Square both sides of the equation: $(x-3) - 2\sqrt{(x-3)(x+1)} + (x+1) = 4$. Collect like terms, and isolate the radical: $x - 3 = \sqrt{(x-3)(x+1)}$. Now square both sides again: $(x-3)^2 = (x-3)(x+1)$. The only value of x that makes these quadratics equal is $x = 3$.

14. If three fair dice are tossed and the product of the numbers that appear is even, what is the probability that the sum of the numbers is also even? Express as a fraction in simplest form.

Solution: In looking at the sum and product, we only need to consider whether each die is even or odd. That leaves 8 equally likely outcomes of the dice roll: EEE, EEO, EOE, OEE, EOO, OEO, OOE, and OOO. If the product of those numbers is even, that means that at least one is even (that is, not OOO). The remaining 7 results are still equally likely, and 4 of those possibilities (EEE, EOO, OEO, and OOE) have an even sum, so the probability is $\frac{4}{7}$.

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15. Find k so that $\dfrac{2x^3+3x^2+kx+3}{x+2} = 2x^2 - x - 6$, remainder 15.

Solution 1: The simplest method is to use the Remainder Theorem, which says that if the polynomial $f(x)$ is divided by $x-a$, the remainder is $f(a)$. With $f(x) = 2x^3+3x^2+kx+3$, that means that $15 = f(-2) = -16+12-2k+3$, so $k = -8$.

Solution 2: Using synthetic division, the remainder is $-2k-1 = 15$, so $k = -8$. (Because synthetic division also serves as an alternate way to evaluate $f(-2)$, this is equivalent to the remainder theorem approach above.)

$$\begin{array}{r|rrrr} -2 & 2 & 3 & k & 3 \\ & & -4 & 2 & -2k-4 \\ \hline & 2 & -1 & k+2 & -2k-1 \end{array}$$

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Using as many of the ten digits (0–9) as needed, but no more than once each, construct the largest possible five-digit odd number with a 5 in the hundreds place.

Solution: If we wish to fill in the blanks in _ _5_ _ to make the largest possible number, we use the largest available digit for each position from left to right: 98573. (For the last digit, we cannot use 4 or 6 because the number is supposed to be odd.)

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2. How many positive factors, including 1 and 2012, does the number 2012 have?

Solution: $2012 = 1 \times 2012 = 2 \times 1006 = 4 \times 503$. 503 is a prime number, so there are no other factorizations; that means there are 6 factors.

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3. Find the measure in degrees of one interior angle of a regular pentagon.

Solution: The sum of the interior angles of an n-gon is $(n-2)180^\circ$, so the five interior angles in any pentagon have total measure 540°. In a regular pentagon, all are the same size: 108°.

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4. $\angle B$ is complementary to $\angle A$, and $\angle C$ is supplementary to $\angle A$. Find $m\angle C - m\angle B$ (in degrees).

Solution: $m\angle A + m\angle B = 90^\circ$, and $m\angle A + m\angle C = 180^\circ$, so $m\angle C - m\angle B = 90^\circ$.

5. A 100-foot ladder leans against a vertical wall. The foot of the ladder is 28 feet from the base of the wall. If the bottom of the ladder is pulled away from the wall 32 more feet, by how many feet does the top of the ladder slip?

Solution: Using the Pythagorean theorem, the top of the ladder is initially $\sqrt{100^2 - 28^2} = 96$ feet up the wall—i.e., the ladder forms a 28-96-100 triangle with the ground and the wall. When the ladder's base is 60 feet away from the wall, the top will be $\sqrt{100^2 - 60^2} = 80$ feet up the wall (a 60-80-100 triangle). This is a drop of <u>16 feet</u>.

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6. If 20% of 12 is the same as 30% of m, find m.

Solution: Solving $(0.2)(12) = (0.3)(m)$ leads to $\underline{m = 8}$.

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7. Find the ratio of (12 yds, 20 in) to (20 yds, 12 in). Express as a fraction in simplest form.

Solution: Express both measurements in a single unit (inches or feet or yards), then simplify: $\frac{12\times 36+20 \text{ inches}}{20\times 36+12 \text{ inches}} = \frac{452}{732} = \frac{113}{183}$

Section B. Each correct answer is worth 2 points.

8. Find $m + n - d$, where m is the mean, n the median, and d the mode for the set of numbers $12, 20, 5, 15, 5, 0, 9, 13, 11$.

Solution: Listing the numbers in order, we have $0, 5, 5, 9, 11, 12, 13, 15, 20$. The sum is 90, so $m = 10$; the median is $n = 11$ and the mode is $d = 5$. Therefore, $m+n-d = 10+11-5 = 16$.

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9. The average of 12 different positive integers is 20. What is the largest possible value of any of these numbers?

Solution: In order for the 12th value to be as large as possible, the other 11 numbers must be as small as possible: $1, 2, 3, \ldots, 11$. Those numbers add up to 66, and the 12th number is $240 - 66 = 174$.

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10. Find p in the Fibonacci-style sequence $20, m, n, 12, p$.

Solution: We have the system $20+m = n, m+n = 12$, and $n+12 = p$. Then $(n-20)+n = 12$, so $n = 16$, and therefore $p = 28$.

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11. Find the point on the number line that is $\frac{2}{3}$ of the distance from -20 to -12.

Solution: -20 and -12 are 8 units apart, so $\frac{2}{3}$ of that distance is $\frac{16}{3}$; $-20 + \frac{16}{3} = -\frac{44}{3} = -14\frac{2}{3}$.

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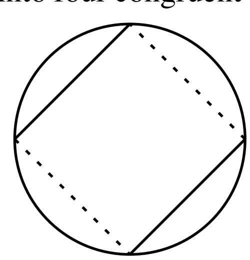
12. In a circle whose diameter is 12 units, two parallel chords divide the circle into four congruent arcs. Find the length of one of these chords.



Solution: The four congruent arcs must each be 90°, so those chords must be like those in the figure on the right (either the two solid chords, or the two dashed chords). Those chords are the sides of a square with diagonal 12 units, so they are $\frac{12}{\sqrt{2}} = 6\sqrt{2} \doteq 8.4853$ units long.

Section C. Each correct answer is worth 3 points.

13. Find the value of c for which the roots of $2x^2 - 21x + c = 0$ are in the ratio $1 : 2$.

Solution: Suppose that a is the smaller root (in absolute value), so the other root is $2a$. Then the left side factors as $2(x-a)(x-2a) = 2(x^2 - 3ax + 2a^2) = 2x^2 - 6ax + 4a^2$. Matching the coefficients, we have $-6a = -21$, so $a = \frac{7}{2}$, and therefore $c = 4a^2 = 49$.

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14. Three solid metal spheres with radii of 3 cm, 4 cm, and 5 cm are melted and recast into a single (solid) sphere. Find the surface area of the new sphere. Express your answer in exact form in terms of π.

Solution: A sphere has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$. The total volume of the three spheres is $\frac{4}{3}\pi(3^3 + 4^3 + 5^3) = \frac{4}{3}\pi \cdot 216 = \frac{4}{3}\pi(6^3)$. Thus the resulting sphere has a radius of 6 cm, and a surface area of $4\pi(6^2) = 144\pi$ square cm.

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15. Find all points of intersection for the circle $(x-3)^2 + (y+2)^2 = 25$ and the ellipse $(x-3)^2 + 10(y+2)^2 = 169$.

Solution: Subtract the first equation from the second, leaving $9(y+2)^2 = 144$. Then $(y+2)^2 = 16$, so $y = -2 \pm 4 = -6$ or 2. Subsituting $(y+2)^2 = 16$ into either equation leaves $(x-3)^2 = 9$, so $x = 3 \pm 3 = 0$ or 6. That leads to the four points $(0, -6)$, $(0, 2)$, $(6, -6)$, and $(6, 2)$.