BLUFFTON
EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2011
ANSWER KEY

PART I

Section A

1. 0
2. 60%
3. midpoint
4. $9m - 31$
5. 25
6. $1, 2$ **(need both)**
7. $\frac{247}{400}$

Section B

8. 8 dollars per rose
9. 100 rectangles
10. 19
11. $\sin\theta$
12. $-\frac{2011}{2012}$

Section C

13. i
14. $(1, 2)$
15. 2035

PART II

Section A

1. 12.62
2. If not Geometry, then not Algebra.
3. $\frac{7}{10}$ *or* 7 : 10 *or* 7 to 10
4. 56%
5. 23
6. Newton *or* Leibniz
7. 1597

Section B

8. $2011\sqrt{3}$ *or* 3483.15 units
9. $x = -3, x = -1, x = 3$ **(need all)**
10. (a) A (b) S
11. $40^\circ, 70^\circ$, or 100° **(need all)**
12. $x = \dfrac{b+c}{2a-b-c}$ *or* $\dfrac{1}{\frac{2a}{b+c} - 1}$

Section C

13. 5 units
14. $-10, -6, -4, 0$ **(need all)**
15. 7.2 *or* $\frac{36}{5}$ units

MATHEMATICS CONTEST 2011
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Simplify: $2^0 - 1^1$

Solution: $2^0 - 1^1 = 1 - 1 = \underline{0}$.

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2. 30 of the 75 algebra students ride the bus to and from school. What percent of the algebra students do not ride the bus?

Solution: The percent who do not ride the bus is $\frac{45}{75} = 0.6 = \underline{60\%}$.

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3. In geometry, what name is given to the point that divides a line segment into two congruent segments?

Solution: The point that divides a segment into two congruent segments is the midpoint.

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4. Subtract $20 + 11m$ from $20m - 11$.

Solution: $20m - 11 - (20 + 11m) = 20m - 11 - 20 - 11m = \underline{9m - 31}$.

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5. Simplify: $\sqrt{25^2}$

Solution: $\sqrt{25^2} = \underline{25}$.

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6. There are two positive integers n for which $(n)(n) = n^n$. Find both of those integers.

Solution: The numbers are 1 and 2: $(1)(1) = 1^1$ and $(2)(2) = 2^2$.

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7. Simplify: $\left(\frac{11}{20} + \frac{3}{20}\right) - \left(\frac{11}{20} \cdot \frac{3}{20}\right)$. Express as a ratio of two integers in simplest form.

Solution: $\left(\frac{11}{20} + \frac{3}{20}\right) - \left(\frac{11}{20} \cdot \frac{3}{20}\right) = \frac{14}{20} - \frac{33}{400} = \frac{280}{400} - \frac{33}{400} = \frac{247}{400}$.

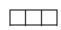
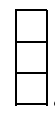
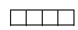
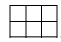
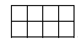
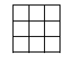
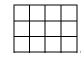
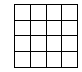
Section B. Each correct answer is worth 2 points.

8. Pete bought 12 roses at \$6.00 each, and Gypsy bought 6 roses at \$12.00 each. What was the average price per rose?

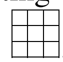
Solution: The average price is $\frac{(12)(\$6)+(6)(\$12)}{12+6} = \frac{\$144}{18} = \underline{\$8.00}$ per rose.

9. How many rectangles (all sizes) are in the 4×4 square grid shown?

<u>Solution 1:</u> There are a total of <u>100</u> rectangles:

16 □, 24  / , 16  / , 8  (etc.), 9 , 12 , 6 , 4 , 4 , 1 

<u>Solution 2:</u> (This is not so much a solution method as an interesting observation.)

In □, there is only 1 rectangle. In , there are 9 rectangles. In , there are 36 rectangles. In general, in an $n \times n$ grid, one can find T_n^2 rectangles, where T_n is the nth triangular number.

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10. If $f(x) = x^2 - 3x + 4$ and $g(x) = 5x - 1$, find $g(f(3))$.

<u>Solution:</u> $f(3) = 4$, so $g(f(3)) = g(4) = \underline{19}$.

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11. $\sin\theta \cdot \cos\theta \cdot \tan\theta \cdot \cot\theta \cdot \sec\theta$ is equal to one of the six trig functions. Which one?

<u>Solution:</u> Because $\tan\theta$ and $\cot\theta$ are reciprocals, as are $\cos\theta$ and $\sec\theta$, the only thing left is $\sin\theta$. That is, $\sin\theta \cdot \cos\theta \cdot \tan\theta \cdot \cot\theta \cdot \sec\theta = \sin\theta \cdot \cos\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} \cdot \frac{1\theta}{\cos\theta} = \underline{\sin\theta}$.

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12. Determine k so that the line with equation $kx + (k+1)y - 20 = 11$ has slope $m = 2011$.

<u>Solution:</u> The slope of this line is $m = -\frac{k}{k+1}$, so $2011(k+1) = -k$, or $k = -\frac{2011}{2012}$.

<u>*Section C*</u>. *Each correct answer is worth 3 points.*

13. Let $i = \sqrt{-1}$. Simplify completely: $\dfrac{20i^{20} + 11i^{11}}{20i^{11} - 11i^{20}}$.

<u>Solution:</u> $i^{20} = (-1)^{10} = 1$ and $i^{11} = i(-1)^5 = -i$, so

$$\frac{20i^{20} + 11i^{11}}{20i^{11} - 11i^{20}} = \frac{20 - 11i}{-11 - 20i} = \frac{20 - 11i}{-11 - 20i} \cdot \frac{11 - 20i}{11 - 20i} = \frac{(20 - 11i)(11 - 20i)}{-11^2 - 20^2} = \frac{-521i}{-521} = i.$$

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14. The graph of $y = \dfrac{x^2 + 2x - 3}{x^2 - 1}$ has a "hole" in it. What are the coordinates of that hole? Write as an ordered pair (x, y).

<u>Solution:</u> $y = \dfrac{x^2 + 2x - 3}{x^2 - 1} = \dfrac{(x+3)(x-1)}{(x+1)(x-1)} = \dfrac{x+3}{x+1}$ for $x \neq 1$, so the hole is at $x = 1$, $y = \frac{1+3}{1+1} = 2$.

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15. In the sequence $2011, 2012, 2013, \ldots$, each term after the third term is found by subtracting the previous term from the sum of the two preceding terms. The fourth term would be $2011 + 2012 - 2013 = 2010$. So the first 4 terms are now $2011, 2012, 2013, 2010$. What is the 25^{th} term?

<u>Solution:</u> Let a_n represent the nth number. Continuing with the pattern, we have $a_5 = 2012 + 2013 - 2010 = 2015$, $a_6 = 2013 + 2010 - 2015 = 2008$, $a_7 = 2010 + 2015 - 2008 = 2017$, $a_8 = 2015 + 2008 - 2017 = 2006, \ldots$. The pattern is: if n is odd, $a_n = a_{n-2} + 2$, and if n is even, $a_n = a_{n-2} - 2$. More usefully, for odd n, $a_n = 2010 + n$, so $a_{25} = 2035$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Find $\sqrt[3]{2011}$ to the nearest hundredth.

Solution: $\sqrt[3]{2011} = 12.62226683\ldots$; rounded to the nearest hundredth, that is 12.62.

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2. Write the contrapositive of the statement "If Algebra, then Geometry."

Solution: The contrapositive of a statement "If P then Q" is "if not Q then not P," so for the given statement, the answer is "If not Geometry, then not Algebra."

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3. Express as a ratio in simplest form: $\dfrac{\text{2 hours, 20 min}}{\text{3 hours, 20 min}}$.

Solution: $\dfrac{\text{2 hours, 20 min}}{\text{3 hours, 20 min}} = \dfrac{\text{140 min}}{\text{200 min}} = \dfrac{7}{10}$. (Note that the units cancel.)

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4. The Blues played 50 games. They won 6 more games than they lost. What percent of the games did they win?

Solution: They won 28 games and lost 22 games, so they won $\frac{28}{50} =$ 56% of their games. (Note: One contest taker pointed out that we should have specified in the question that there were no ties.)

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5. If 2011 is divided by n, the quotient is 87 and the remainder is 10. Find the value of n.

Solution 1: If $2011 \div n = 87$ rem 10, then $2011 = 87n + 10$, or $2001 = 87n$, so $n = 23$.

Solution 2: By trial and error: $2011 \div 87 = 23.1149\ldots$, so the (whole number) quotient is 23.

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6. Name either of the two independent inventors of calculus. (Last name is enough.)

Solution: Calculus was invented by Isaac Newton and Gottried Leibniz.

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7. The Fibonacci Sequence is the numbers 1, 1, 2, 3, 5, 8, 13, 21, ... What is the greatest number in this sequence that is still less than 2011?

Solution 1: The sequence continues: 34, 55, 89, 144, 233, 377, 610, 987, 1597, ...

Solution 2: (This is not a practical approach, but includes some useful information about the Fibonacci numbers.) *Binet's formula* tells us that the nth Fibonacci number is $F_n = (\varphi^n - \varphi^{-n})/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. For large n, this means that $F_n \approx \varphi^n/\sqrt{5}$. Solving $F_n < 2011$ then roughly corresponds to solving $\varphi^n < 2011\sqrt{5}$, so $n < \log_\varphi(2011\sqrt{5}) = 17.479\ldots$, so we want $F_{17} \approx \varphi^{17}/\sqrt{5} = 1596.99\ldots$, or 1597.

Section B. Each correct answer is worth 2 points.

8. In a 30-60-90 right triangle, the longest side and the shortest side differ by 2011. What is the length of the longer leg? The answer may be expressed exactly, or rounded to the nearest hundredth.

Solution: In a 30-60-90 right triangle, if the shorter leg has length a, then the other leg has length $b = a\sqrt{3}$ and the length of the hypotenuse is $c = 2a$. It was given that $c - a = 2011 = a$, so $b = 2011\sqrt{3}$ or about 3483.15.

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9. Solve for all real values of x: $x^3 - 9 = 9x - x^2$

Solution: Group terms and factor: $0 = x^3+x^2-9x-9 = x^2(x+1)-9(x+1) = (x^2-9)(x+1)$, so $x = -3$, $x = -1$, or $x = 3$.

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10. Answer each of the following with A (Always), S (Sometimes), or N (Never). (Both answers must be correct to receive credit.)
    (a) Two lines parallel to a third line are A-S-N parallel to each other.
    (b) Two lines perpendicular to a third line are A-S-N perpendicular to each other.

Solution: (a) Two lines parallel to a third line are always parallel to each other. (b) This statement is never true in the plane, but is sometimes true in three dimensions. In particular, it is true for the x-, y-, and z-axes.

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11. $\triangle ABC$ is isosceles. If $m\angle A = 40^\circ$, find all possible values of $m\angle B$.

Solution: If $\angle A$ is one of the base angles, then the other angles measure 40° and 100°. If $\angle A$ is not a base angle, then the other angles both measure 70°.

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12. Solve for x in terms of a, b, and c: $ax - c = bx + x(c - a) + b$

Solution: Collect like terms: $ax - bx - x(c - a) = c + b$, so $x(2a - b - c) = b + c$, or $x = \dfrac{b+c}{2a-b-c}$. (Alternatives: $\dfrac{1}{\frac{2a}{b+c} - 1}$ or $\dfrac{-b-c}{b+c-2a}$ or ...)

Section C. Each correct answer is worth 3 points.

13. Given the ellipse $16x^2 + 9y^2 - 54y + 64x + 1 = 0$, find the exact distance between the uppermost point of this ellipse and the leftmost point of this ellipse.

Solution 1: The distance we seek is the length of the hypotenuse of a triangle, where the legs are the ellipse's semi-major and semi-minor axes. To find the lengths of those legs, collect like terms on the left side of the equation, then complete the squares:

$$16(x^2 + 4x) + 9(y^2 - 6y) + 1 = 16(x^2 + 4x + 4) + 9(y^2 - 6y + 9) + 1 - 64 - 81$$
$$= 16(x+2)^2 + 9(y-3)^2 - 144$$

In standard form, the equation is $\frac{1}{9}(x+2)^2 + \frac{1}{16}(y-3)^2 = 1$, meaning that the semi-major and -minor axes have lengths 4 and 3, and the hypotenuse has length 5.

Solution 2: Use the quadratic formula to solve the given equation for y: $9y^2 - 54y + (16x^2 + 64x + 1) = 0$, so

$$y = \frac{54 \pm \sqrt{54^2 - 36(16x^2 + 64x + 1)}}{18} = 3 \pm \frac{1}{3}\sqrt{80 - 16x^2 - 64x}$$

Plotting the "+" equation allows one to observe that the uppermost point of the ellipse is $(-2, 7)$ and the leftmost point is $(-5, 3)$, and the distance between those points is 5.

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14. List all integers m, $-10 \le m \le 10$, such that $P = m^2 - 20m + 11$ is a prime number.

Solution: Let $f(m) = m^2 - 20m + 11$. For m odd, $f(m)$ is even (and never equals 2), so we can immediately remove those values of m from consideration. Furthermore, $f(m) < 0$ for $1 \le m \le 10$ (recall that a prime number must be positive). That leaves us with only six values to check, of which four yield prime numbers: $f(-10) = 311$, $f(-6) = 167$, $f(-4) = 107$, $f(0) = 11$.

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15. In the figure shown, chord $\overline{MU}$ bisects chord $\overline{BT}$. The length of tangent $\overline{MA}$ is 8 and $AT = 4$. If $HU = 5$, find the length of $\overline{MH}$.

Solution: Using a theorem about secants and tangents, $AM^2 = AT \cdot AB$, so $AB = 16$, and therefore $TB = 12$. A similar theorem about chords says that $TH \cdot HB = MH \cdot HU$. With $TH = HB = \frac{1}{2}TB = 6$ and $HU = 5$, we have $MH = \frac{36}{5} = 7.2$.