BLUFFTON
EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2010
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Solve for x: $3x + 9 = 2010$

Solution: $3x + 9 = 2010 \iff 3x = 2001 \iff x = \underline{667}$.

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2. When a 6-faced die (number cube) is rolled, what is the probability of rolling an even number?

Solution: Three of the six faces have even numbers, so $P(\text{even}) = \frac{3}{6} = \frac{1}{2} = \underline{0.5}$.

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3. The perimeter of isosceles $\triangle MAT$ is 32 units. If one of the congruent legs is 12 units, how long is the base?

Solution: If x is the length of the base, then the perimeter is $12 + 12 + x = 32$, so $x = \underline{8 \text{ units}}$.

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4. Three consecutive odd integers are added. If the smallest of these three integers is $2k - 1$, then how can the sum be represented in terms of k?

Solution: If the smallest of the three integers is $2k - 1$, then the sum is $(2k - 1) + (2k + 1) + (2k + 3) = \underline{6k + 3}$.

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5. What is the slope of the line $x + 2y = 5 - 2x$?

Solution: $x + 2y = 5 - 2x \iff 2y = 5 - 3x \iff y = 2.5 - 1.5x$, so the slope is $\underline{-1.5}$.

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6. What is the name of the point $(0, 0)$ where the x-axis and the y-axis intersect?

Solution: The point $(0, 0)$ is the <u>origin</u>.

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7. Let $n = -3$. Evaluate: $-5^2 + 5n + n^2$

Solution: $-5^2 + 5n + n^2 = -25 + 5(-3) + (-3)^2 = -25 - 15 + 9 = \underline{-31}$.

Section B. Each correct answer is worth 2 points.

8. Find the area of a trapezoid with vertices $A(3, 2)$, $B(7, 6)$, $C(10, 6)$, and $D(10, 2)$.

Solution 1: Let E be the point $(7, 2)$. Then this trapezoid can be partitioned into isosceles right $\triangle ABE$ with leg length 4 (area 8 square units), and 3×4 rectangle $BCDE$ (area 12 square units), for a total of <u>20 square units</u>.

Solution 2: This trapezoid hasparallel sides of length $\ell_1 = AD = 7$ and $\ell_2 = BC = 3$, and height $h = CD = 4$. The formula for the area of a trapezoid is $\frac{1}{2}(\ell_1 + \ell_2)h = \frac{1}{2}(7 + 3)(4) = 20$ square units.

9. Simplify and write your answer using only positive exponents: $\dfrac{2a^{-1}b^2}{3^{-1}a^2bc^0}$

Solution: $\dfrac{2a^{-1}b^2}{3^{-1}a^2bc^0} = \dfrac{2 \cdot 3b^2}{a^3b} = \dfrac{6b}{a^3}$

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10. Evaluate: $\dfrac{6! - 5!}{4! + 3!}$

Solution: $\dfrac{6! - 5!}{4! + 3!} = \dfrac{720 - 120}{24 + 6} = \dfrac{600}{30} = \underline{20}$.

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11. Simplify: $5\left[7 - 2^2(6 - 2 \cdot 4) + 5\right] - 5 \cdot 1^3 + 2010^0$

Solution: $5\left[7 - 2^2(6 - 2 \cdot 4) + 5\right] - 5 \cdot 1^3 + 2010^0 = 5[7 - 4(6 - 8) + 5] - 5 + 1 = 5[12 - 4(-2)] - 4 = 5[12 + 8] - 4 = \underline{96}$.

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12. In pentagon *ARITH*, $m\angle A = 125^\circ$ and $m\angle R = 135^\circ$. The measures of the three remaining angles are in the ratio of 1 to 2 to 4. Find the measure of the largest angle of that pentagon.

Solution: The interior angles of a pentagon sum to 540°. The two given angles account for 260°, so the remaining three angles sum to 280°. For the ratio $1 : 2 : 4$, those angles are 40°, 80°, and $\underline{160^\circ}$.

Section C. Each correct answer is worth 3 points.

13. Factor completely: $4x^4 + 16x^2y^2 + 25y^4$

Solution: Complete the square for the first and last terms:

$$\begin{aligned}(4x^4 + 25y^4) + 16x^2y^2 &= \left[(2x^2)^2 + 20x^2y^2 + (5y^2)^2\right] - 20x^2y^2 + 16x^2y^2 \\ &= (2x^2 + 5y^2)^2 - 4x^2y^2.\end{aligned}$$

This is a difference of squares, so it factors into $(2x^2 + 2xy + 5y^2)(2x^2 - 2xy + 5y^2)$.

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14. Simplify completely so that no radical remains in the denominator: $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{27} - \sqrt{3}}$

Solution: $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{27} - \sqrt{3}} = \dfrac{\sqrt{3} - \sqrt{2}}{3\sqrt{3} - \sqrt{3}} = \dfrac{\sqrt{3} - \sqrt{2}}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{3 - \sqrt{6}}{6} = \dfrac{1}{2} - \dfrac{\sqrt{6}}{6}$.

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15. The sum of the digits of a three-digit number is 12. How many such three-digit numbers are there? (Note: a three-digit number does not begin with 0.)

Solution: If the first digit is k, the other two digits add to $12 - k$. For $k = 1$, there are 8 such numbers (129, 138, ..., 183, 192), and for $k = 2$, there 9 such numbers (219, 228, ..., 282, 291). For $k = 3$, there are 10 (309, 318, ..., 381, 390), and following a similar pattern, there are $13 - k$ for $3 \le k \le 9$. That gives a total of $8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 = 66$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Find $3\sqrt{2010}$ to the nearest hundredth.

Solution: $3\sqrt{2010} = 134.4990706\ldots$, which rounds to $\underline{134.50}$.

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2. At what point (x, y) does the line $y = 3x + 6$ cross the x-axis?

Solution: Solving $0 = 3x + 6$ yields $x = -2$, so the line crosses the x-axis at $\underline{(-2, 0)}$.

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3. Write an algebraic expression for this verbal expression: the sum of 4 times k and the square of m.

Solution: "The sum of 4 times k and the square of m" means "the sum of $(4k)$ and (m^2)," or $\underline{4k + m^2}$.

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4. 5 is what percent of 8?

Solution: $\frac{5}{8} = 0.625 = \underline{62.5\%}$.

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5. Find the distance between the two points $(3, 12)$ and $(-1, 4)$
   (a) in exact, simplified form, and (b) rounded to the nearest hundredth.

Solution: $d = \sqrt{(3+1)^2 + (12-4)^2} = \sqrt{16 + 64} = \sqrt{80} = \sqrt{(16)(5)} = \underline{4\sqrt{5}} \doteq \underline{8.94}$.

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6. The area of a square is 2010 square units. Find the length of its diagonal to the nearest whole number.

Solution: If $A = 2010 = s^2$, then the side length is $s = \sqrt{2010}$, and the length of the diagonal is $s\sqrt{2} = 63.403\ldots$, which rounds to $\underline{63}$.

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7. List all the prime numbers between 90 and 100, inclusive.

Solution: The only prime number between 90 and 100 is $\underline{97}$. ($91 = 7 \times 13$ and $93 = 3 \times 31$.)

Section B. Each correct answer is worth 2 points.

8. How much water must be added to 10 gallons of lemonade to reduce its lemon content from 35% to 20%?

Solution: 10 gallons of lemonade with a 35% lemon content contains 3.5 gallons of lemon juice (and 6.5 gallons of water). We need to add x gallons of water, where $\frac{3.5}{10+x} = 0.2$. Then $3.5 = 2 + 0.2x$, so $x = \underline{7.5 \text{ gallons}}$.

9. $f(x) = x^2 - 3x + 2010$. Find and express your answer in simplest form:
   (a) $f(0)$ (b) $f(-2)$ (c) $f(m)$

Solution: (a) $f(0) = 0^2 - 3(0) + 2010 = \underline{2010}$. (b) $f(-2) = (-2)^2 - 3(-2) + 2010 = 4 + 6 + 2010 = \underline{2020}$. (c) $f(m) = \underline{m^2 - 3m + 2010}$.

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10. Find the sum of all the factors of 2010.

Solution 1: By (careful) trial and error, we can find all pairs of numbers with a product of 2010: $2010 = 1\times 2010 = 2\times 1005 = 3\times 670 = 5\times 402 = 6\times 335 = 10\times 201 = 15\times 134 = 30\times 67$. The sum of these factors is $\underline{4896}$.

Solution 2: The prime factorization of 2010 is $2 \times 3 \times 5 \times 67$, so the factors of 2010 are all numbers that can be formed by multiplying 0 or 1 of each of these prime factors; that is, there are 16 factors, each corresponding to $2^a 3^b 5^c 67^d$, where each exponent is either 0 or 1. This approach leads (of course) to the same list of factors as in the first solution.

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11. The area of a circle is 2010 square units. Find the length of its diameter to the nearest whole number.

Solution: If $A = 2010 = \pi r^2$, then the radius is $r = \sqrt{2010/\pi}$, and the length of the diameter is $2r = 50.588\ldots$, which rounds to $\underline{51}$.

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12. The polynomial equation $P(x) = x^5 + 2x^4 + 5x^3 + 34x^2 + 30x = 0$ has five roots. One of them is $1 + 3i$. Find the other four roots.

Solution 1: Because $P(x)$ has real coefficients, the complex root $1 + 3i$ must be matched with its complex conjugate $1 - 3i$. By inspecting the graph of $P(x)$, we see that it crosses the x-axis at the other three (real) zeros: $-3, -1$, and 0.

Solution 2: Noting that $P(x) = x(x^4 + 2x^3 + 5x^2 + 34x + 30)$, we see that 0 is another root. Observing (as in solution 1) that $1-3i$ is a root, we know that $P(x)$ is divisible by x, $x-1-3i$, and $x - 1 + 3i$. Carrying out those three divisions (synthetically or via long division), we find that $P(x) = x(x - 1 - 3i)(x - 1 + 3i)(x^2 + 4x + 3)$, and the latter quadratic factor equals 0 when $x = -3$ or $x = -1$.

Section C. Each correct answer is worth 3 points.

13. Maternity nurse Obie LaMaz has treated 2010 patients, who were mothers and their newborns. If the only multiple births were 12 sets of twins and 4 sets of triplets, how many mothers were there?

Solution: Let m be the number of mothers. Then the number of newborns is $m + 12 + 8$ (one infant per mother, plus 12 additional infants for the 12 sets of twins, plus 4×2 infants for the 4 set of triplets). Then the total number of patients is $m + (m + 20) = 2m + 20 = 2010$, so $\underline{m = 995 \text{ mothers}}$.

14. A lattice point (a, b) in the coordinate plane is a point in which both a and b are integers. How many lattice points lie in the interior of the circle $x^2 + 6x + y^2 - 4y = 3$?

Solution: Completing the squares leaves

$$(x^2 + 6x + 9) - 9 + (y^2 - 4y + 4) - 4 = 3, \text{ or}$$
$$(x + 3)^2 + (y - 2)^2 = 16.$$

This is a circle centered at $(-3, 2)$ with radius 4. It is useful to observe that the number of lattice points in the interior of this circle is the same as the number in the congruent circle centered at the origin; the latter list includes $(0, 0)$, twelve other points in the axes, and eight points in each quadrant, for a total of 45 lattice points.

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15. A flagpole is situated at the edge of a roof on the top of a building. From an observation point 2010 feet from the base of that building, the angles of elevation of the top and bottom of that flagpole are 14° and 12°, respectively. To the nearest foot, find the length (height) of the flagpole.

Solution: Let x the building's height, and h the flagpole's height. Then $x = 2010 \tan 12^\circ$ and $x + h = 2010 \tan 14^\circ$, so $h = 2010(\tan 14^\circ - \tan 12^\circ) \doteq 73.91$ feet, which rounds to 74 feet.