MATHEMATICS CONTEST 2009
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. List all the prime numbers between 20 and 30.

Solution: Recall that n is a prime number if it has exactly two factors: 1 and n. There is only one even prime (2), and of the odd numbers between 20 and 30, $21 = 3 \times 7$, $25 = 5 \times 5$, and $27 = 3 \times 9$. That leaves 23 and 29.

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2. *Perimeter* is to *square* as ________ is to *circle*.

Solution: Perimeter is the distance around a square (or any polygon); circumference is the name we give to the distance around a circle.

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3. Find the sum: $1 + 1 + 2 + 6 + 11 + 20 + 37 + 68 + (-58) + (-27) + (-10) + (-1)$

Solution: The first four numbers sum to 10. Of those that are left, each of the four positive numbers pairs with one of the four negative numbers to give 10—for example, $11 + (-1) = 10$, $20 + (-10) = 10$, etc. That gives five 10s, or 50.

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4. Which of the following capital letters appear to have a horizontal line of symmetry?

A B C D E F G H I J K L M N O P

Solution: A horizontal line of symmetry means that the bottom half is a mirror image of the top half. That describes the letters **B C D E H I K O**.

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5. If a Sunday falls on the 24th of the month, what is the probability that the *next* Sunday will fall on the 31st of that month? Express your answer as a fraction or ratio in simplest form.

Solution: Seven months (January, March, May, July, August, October, and December) have 31 days, so this probability is 7/12.

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6. Darryl scored 36 points. Steve scored 12 points more than did Don, who had 1/4 as many points as did Darryl. How many total points did all three players score?

Solution: Darryl had 36 points, Don had $\frac{1}{4} \times 36 = 9$ points, and Steve had $9 + 12 = 21$ points, giving a total of 66 points.

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7. Evaluate: $2009^1 + 2009^0 - \dfrac{30}{1/3}$.

Solution: $2009^1 + 2009^0 - \dfrac{30}{1/3} = 2009 + 1 + 90 = \underline{1920}$.

Section B. Each correct answer is worth 2 points.

8. Find the *slope* of the line $2x + y - \frac{3}{4} = \frac{4}{3}$.

Solution: After rearranging terms, we have $y = -2x + b$. (It is not necessary to combine the two fractions; we are only interested in the slope.) So, the slope is -2.

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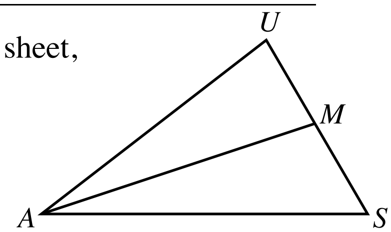
9. Shown below are the *first two steps* in a proof. **O**n your answer sheet, fill in the missing statement for step 2.

Given: $\triangle USA$ with M the midpoint of $\overline{US}$.

Prove: Area of $\triangle MAU$ = Area of $\triangle MAS$.



Statements	Reasons
1. M is midpoint of $\overline{US}$	1. Given
2. ________________	2. Definition of midpoint

Solution: While the definition of "midpoint" says a number of things, the crucial detail from the definition—the fact that allows us to make progress toward proving that the triangles have equal area—is that $SM = MU$, or equivalently, $\overline{SM} \cong \overline{MU}$.

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10. *How many* of the integers x, $1 \le x \le 500$, do *not* contain the digit "3"?

Solution 1: In order to form all sequences of digits abc (possibly starting with 0) which correspond to these values of x, note that there are 4 possible values for a (0,1,2,4), 9 possible values for b, and 9 possible values for c. That makes a total of $4 \times 9 \times 9 = 324$ possible sequences.

Solution 2: Take out the 100 integers from 300 to 399, then the 4×10 numbers 30–39, 130–139, 230–239, and 430–439, then the 4×9 numbers of the form $ab3$, where $a \in \{0, 1, 2, 4\}$ and $b \in \{0, 1, 2, 4, 5, 6, 7, 8, 9\}$. That leaves $500 - 100 - 40 - 36 = 324$.

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11. Express in simplest radical form (no radicals in the denominator): $\dfrac{3\sqrt{2}}{2\sqrt{3}}$

Solution: Multiply and divide by $\sqrt{3}$: $\dfrac{3\sqrt{2}}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{3\sqrt{2}\sqrt{3}}{2 \cdot 3} = \dfrac{\sqrt{6}}{2}$ or $\dfrac{\sqrt{2}\sqrt{3}}{2}$

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12. Find the *remainder* when $(2x^2 - 2x + 25)$ is divided by $(x - 5)$.

Solution 1: According to the Remainder Theorem, the remainder when a polynomial $p(x)$ is divided by $x - k$ is $p(k)$. In this case, $2(5)^2 - 2(5) + 25 = 65$.

Solution 2: Carrying out the division (synthetically or the "long" way) yields the same remainder.

Section C. Each correct answer is worth 3 points.

13. Let P be the *product* of a collection of two or more positive integers (repeats allowed) which have a *sum* of 13. Find the largest possible value of P.

Solution: The largest possible product is 108 (either $3 \times 3 \times 3 \times 4$ or $3 \times 3 \times 3 \times 2 \times 2$). In general, if the collection of integers must add up to n, the largest product $P_{\max}$ is found by taking k 3's, so that $n - 3k$ equals 0, 2, or 4. That is, if n is a multiple of 3, then $P_{\max} = 3^{n/3}$. If n is one more than a multiple of 3, then $P_{\max} = 3^{(n-4)/3} \times 4$. If n is two more than a multiple of 3, then $P_{\max} = 3^{(n-2)/3} \times 2$.

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14. Given: $\log_b a + \log_b c = b$. Solve for c in terms of a and b without "log" in the answer.

Solution: Using the product rule for logarithms, the left side equals $\log_b(ac)$. Exponentiating both sides of the equation (with base b) gives $ac = b^b$, so $\underline{c = b^b/a}$.

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15. Find the vertex of the parabola $4x^2 + 12x + 2y = -7$. Express as an ordered pair (x, y).

Solution 1: Complete the square: $4(x^2 + 3x + 1.5^2) - 4 \times 1.5^2 + 2y = -7$. Rearranging terms gives $2y = -4(x + 1.5)^2 + 2$, or $y = -2(x + 1.5)^2 + 1$. That places the vertex at $\underline{(-1.5, 1)}$.

Solution 2: Rearrange terms to get $y = -2x^2 - 6x - 3.5$. This is now in "standard form" $(ax^2 + bx + c)$, so the x-coordinate of the vertex is $-\frac{b}{2a} = -\frac{6}{4} = -\frac{3}{2}$. The y-coordinate is therefore $-2(-1.5)^2 - 6(-1.5) - 3.5 = 1$.

Solution 3: Using calculus, we find the derivative of $y = -2x^2 - 6x - 3.5$, which is $y' = -4x - 6$. This equals 0 when $x = -1.5$; that is the x-coordinate of the vertex. Continue as in Solution 2.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Latin teacher Mathematicus had 1 penny, 5 nickels, 10 dimes, 25 quarters, 50 50-cent pieces, and 100 dollar bills. How much money did he have in all?

Solution: He has $1 \times 1 + 5 \times 5 + 10 \times 10 + 25 \times 25 + 50 \times 50 + 100 \times 100 = 13251$ cents, or $\underline{\$132.51}$.

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2. Find $2009\sqrt{2009}$ to the nearest *hundredth*.

Solution: A calculator gives $2009\sqrt{2009} \doteq 90047.13615\ldots$, which rounds to $\underline{90047.14}$.

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3. The greatest common factor of x and y is 3, and their lowest common multiple is 66. x is 33. Find the value of y.

Solution: For any integers x and y, $xy = \text{GCF}(x, y) \times \text{LCM}(x, y)$. Therefore, $33y = (3)(66)$, so $\underline{y = 6}$.

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4. Express as an improper fraction in simplest form: $\frac{2}{5} + 1.\overline{3}$.

Solution: $\frac{2}{5} + 1.\overline{3} = \frac{2}{5} + \frac{4}{3} = \frac{6+20}{15} = \frac{26}{15}$.

5. The numbers 1, 3, 6, 10, 15, 21, ... are known as triangular numbers. What is the largest triangular number *less than 300?*

Solution 1: This can be found by "brute force," by simply adding consecutive integers until 300 is reached, at 276.

Solution 2: The formula for the nth triangular number is $T_n = \frac{n(n+1)}{2}$, so we want $n(n+1) < 600$. That means that n should be close to $\sqrt{600} \doteq 24.5$. Noting that $T_{24} = 300$, we know that the answer must be $T_{23} = 276$.

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6. Find the area of a rectangle if its length is 24 and its diagonal has length 25.

Solution: Call the width of the rectangle w. By the Pythagorean theorem, $24^2 + w^2 = 25^2$, so $w = 7$, and the area is $(7)(24) = 168$ square units.

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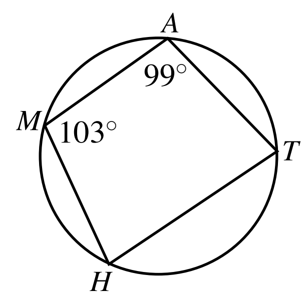
7. Among the adult residents of Mathville, half of the men are married to women, and $\frac{1}{3}$ of the women are married to men. (No one is married to more than one person.) If there are 240 men in the village, how many total adults live in Mathville?

Solution: 120 of the 240 men are married to women, so 120 must be $\frac{1}{3}$ of the women. That means there are 360 women, for a total of $240 + 360 = 600$ adults.

Section B. Each correct answer is worth 2 points.

8. Quadrilateral *MATH* is inscribed in the circle as shown. Name the smallest angle in the quadrilateral and give its measure.



Solution: Quadrilateral *MATH* is called a *cyclic quadrilateral,* meaning that its vertices lie on a circle. Cyclic quadrilaterals have the property that opposite angles are supplementary, so the smallest angle is $\angle T$, with measure 77°.

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9. I bought 3 books at \$19.90 each. My total cost, including sales tax, was \$63.58. What was the sales tax rate (to the nearest half of one percent)?

Solution: Before tax, the books cost \$59.70. If the tax rate is r, then $59.70(1 + r) = 63.58$ (after rounding the right side to the nearest penny). Therefore, $1 + r \doteq \frac{63.58}{59.70} = 1.06499\ldots$, so $r = 0.065 = 6.5\%$.

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10. Let $z = |-x| - |2y| + xy$. Evaluate z if
(a) $x = 3$ and $y = -2$. (b) $x = -3$ and $y = -4$. (For credit, both answers must be correct.)

Solution: (a) With $x = 3$ and $y = -2$, $z = |-3| - |-4| + (3)(-2) = 3 - 4 - 6 = -7$.
(b) With $x = -3$ and $y = -4$, $z = |3| - |-8| + (-3)(-4) = 3 - 8 + 12 = 7$.

11. Larry, Curly, and Moe divide a pile of n pennies like so: If n is **e**ven, Larry takes half the pile. If n is **o**dd, Moe takes one, then Curly takes half of those that remain. This process is repeated until the pile is gone. If the original pile contains 2009 pennies, how many will *Curly* get?

Solution: The table on the right shows the entire process of dividing the pile of pennies. At the end, Curly has 1217 pennies.

Had they started with 2000 pennies, Curly would only have ended with 88 pennies. Starting with 2007 pennies, Curly would have collected 1842 pennies.

Pennies	Larry	Moe	Curly
2009	–	1	1004
1004	502	–	–
502	251	–	–
251	–	1	125
125	–	1	62
62	31	–	–
31	–	1	15
15	–	1	7
7	–	1	3
3	–	1	1
1	–	1	–
	784	8	1217

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12. The base of an isosceles triangle is 25 cm long, and the vertex angle measures 40°. Find the length (in cm) of a leg. (If appropriate, round to two decimal places.)

Solution 1: Call this leg length L. Draw an altitude from the vertex (which bisects the vertex angle), creating a right triangle with an acute angle measuring 20° opposite a side of length 12.5 cm. The hypotenuse of this right triangle has length L, so $\sin 40^\circ = \frac{12.5}{L}$, meaning that $L = \frac{12.5}{\sin 20^\circ} = 36.5475\ldots$ cm, which rounds to 36.55 cm

Solution 2: The base angles of the triangle measure 70°, so by the law of sines, $\frac{25}{\sin 40^\circ} = \frac{L}{\sin 70^\circ}$. Then $L = \frac{25 \sin 70^\circ}{\sin 40^\circ} = 36.5475\ldots$ cm, as before.

Section C. Each correct answer is worth 3 points.

13. Find all solutions (both real and complex) to $x^5 - 14x^3 - 72x = 0$. Express your answers either exactly or rounded to two decimal places.

Solution: The solution set is $\{0, \pm 2i, \pm 3\sqrt{2}\}$.

Clearly, $x = 0$ is a solution. Pulling out a factor of x, we have $x(x^4 - 14x^2 - 72) = 0$. Viewing the fourth-degree factor as a quadratic expression in x^2, we have

$$x^2 = \frac{14 \pm \sqrt{14^2 - 4(1)(-72)}}{2} = 7 \pm 11 = 18 \text{ or } -4.$$

$x^2 = -4$ gives the solutions $x = \pm 2i$, and $x^2 = 18$ gives the solutions $x = \pm\sqrt{18} = \pm 3\sqrt{2}$.

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14. The length of each side of an isosceles triangle is an integer. The legs have length $x + 1$, and the base length is $3x - 2$. Determine all possible values for this triangle's *perimeter.*

Solution: In a triangle, the sum of any two side lengths must exceed the third, so $2x + 2 > 3x - 2$ and $4x - 1 > x + 1$. The first inequality implies that $x < 4$, while the second says that $3x > 2$. Combined with the fact that all side lengths are integers, we see that x must be 1, 2, or 3. The perimeter will be $2(x + 1) + 3x - 2 = 5x$, or 5, 10, 15.

15. Evaluate: $\sum_{k=1}^{6} \left[k^2 + (k-1)!\right]$.

Solution: It is helpful (though not necessary) to split the sum into two parts:

$$\begin{aligned}
\sum_{k=1}^{6} \left[k^2 + (k-1)!\right] &= \sum_{k=1}^{6} k^2 + \sum_{k=1}^{6} (k-1)! \\
&= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) + (0! + 1! + 2! + 3! + 4! + 5!) \\
&= (1 + 4 + 9 + 16 + 25 + 36) + (1 + 1 + 2 + 6 + 24 + 120) \\
&= 91 + 154 \\
&= \underline{245}.
\end{aligned}$$