BLUFFTON

EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2008
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. How many positive integral factors does 10 have?

Solution: The positive integral factors of 10 are $1, 2, 5$, and 10—there are four factors.

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2. Give the more common name for a regular quadrilateral.

Solution: "Regular" means congruent angles and congruent sides. A quadrilateral has four sides, so a regular quadrilateral is a square.

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3. Find the arithmetic mean (average) of the first seven positive integers.

Solution: $1+2+3+4+5+6+7=28$; the average is $\frac{28}{7}=\underline{4}$.

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4. Which one is not equal to the other four? Answer with a **c**apital letter.

(A) $(20)(2.4)$ **(B)** 60% of 80 **(C)** $(2(4+8))3$
(D) $36/(3/4)$ **(E)** $2\left(\sqrt{25+75}+\sqrt{\sqrt{81}}\right)+(12\cdot 2-2)$

Solution: Expression C is equal to 72; the others are equal to 48.

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5. Solve for x: $4x+1=3(x+1)$.

Solution: Multiply out the right side: $4x+1=3x+3$. Then $\underline{x=2}$.

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6. Simplify: $10^3+30^2+10^2+2^3$.

Solution: $10^3+30^2+10^2+2^3=1000+900+100+8=\underline{2008}$.

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7. The clock on the wall is a standard clock with an hour hand and a minute hand. What is the measure (in degrees) of the smaller angle formed at 7:00?

Solution: There are $360°/12=30°$ between each numeral on the clock face, so from 7 to 12, the angle is $\underline{150°}$.

Section B. Each correct answer is worth 2 points.

8. Write the numerical value of $\log_2 8$.

Solution: If $\log_2 8 = x$, then $8 = 2^x$, so $x = \underline{3}$. ($\log_2 8$ asks the question, "To what power must we raise 2 in order to get 8?")

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9. Subtract the sum of $(n^2 + 3n - 1)$ and $(-2n^2 + 1)$ from $(5n^2 - 4n + 3)$.

Solution: $(n^2 + 3n - 1) + (-2n^2 + 1) = -n^2 + 3n$; subtracting this from $(5n^2 - 4n + 3)$ gives $\underline{6n^2 - 7n + 3}$.

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10. Express as the ratio of two integers in simplest form: $0.4\overline{3} - 0.3\overline{4}$.

Solution 1: Note that $0.4\overline{3} = 0.1 + 0.\overline{3} = \frac{1}{10} + \frac{1}{3}$ and $0.3\overline{4} = -0.1 + 0.\overline{4} = -\frac{1}{10} + \frac{4}{9}$. Therefore,

$$0.4\overline{3} - 0.3\overline{4} = \frac{1}{10} + \frac{1}{3} + \frac{1}{10} - \frac{4}{9} = \frac{1}{5} + \frac{1}{3} - \frac{4}{9} = \frac{9}{45} + \frac{15}{45} - \frac{20}{45} = \frac{4}{45}.$$

Solution 2: Let $x = 0.4\overline{3}$ and $y = 0.3\overline{4}$. Then

multiply by 100:	$100x = 43.\overline{3}$	$100y = 34.\overline{4}$
multiply by 10:	$10x = 4.\overline{3}$	$10y = 3.\overline{4}$
subtract:	$90x = 39$	$90y = 31$

so $x - y = \frac{39-31}{90} = \frac{8}{90} = \frac{4}{45}$.

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11. If $\begin{vmatrix} a & b \\ 3 & 4 \end{vmatrix} = 12$, solve for b in terms of a. (*Note:* $\begin{vmatrix} r & s \\ t & u \end{vmatrix}$ is the <u>determinant</u> of that matrix.)

Solution: The given equation simplifies to $4a - 3b = 12$, so $3b = 4a - 12$, and therefore $b = \frac{4}{3}a - 4$.

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12. Given isosceles $\triangle MNO$, if base $\overline{MO}$ is $\frac{1}{2}$ the length of $\overline{MN}$, and the perimeter of $\triangle MNO$ is 100 cm, find the length of $\overline{ON}$.

Solution: If $\overline{MO}$ is the base, then $\overline{MN}$ and $\overline{ON}$ are congruent. Let x be the common length of $\overline{ON}$ and $\overline{MN}$. We have $x + x + \frac{1}{2}x = 100$ cm. Solving $2.5x = 100$ cm gives $x = \underline{40\text{ cm}}$.

Section C. Each correct answer is worth 3 points.

13. Express as a single fraction in simplest form: $(ab^{-1} + ba^{-1})^{-1}$.

Solution: $(ab^{-1} + ba^{-1})^{-1} = \left(\frac{a}{b} + \frac{b}{a}\right)^{-1} = \left(\frac{a^2 + b^2}{ab}\right)^{-1} = \frac{ab}{a^2 + b^2}$

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14. If $i = \sqrt{-1}$, express $\dfrac{i^{2007} + i^{2008}}{i^{2008} + i^{2009}}$ in simplest form.

Solution 1: Factor and cancel out a common factor of i^{2007} from the numerator and denominator:

$$\frac{i^{2007} + i^{2008}}{i^{2008} + i^{2009}} = \frac{i^{2007}(1 + i)}{i^{2008}(1 + i)} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i.$$

Solution 2: Powers of i go in cycles of 4: $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$, and then we start over again. If n is a multiple of 4, $i^n = 1$. Therefore, for example, $i^{2007} = i^{2004+3} = i^3 = -i$, so

$$\frac{i^{2007} + i^{2008}}{i^{2008} + i^{2009}} = \frac{-i+1}{1+i} = \left(\frac{1-i}{1+i}\right)\left(\frac{1-i}{1-i}\right) = \frac{(1-i)^2}{2} = \frac{1-2i-1}{2} = -i.$$

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15. Factor completely over the rational numbers: $m^4 + 2m^2n^2 + 9n^4$.

Solution: Working with the first and last terms, we complete the square:

$$m^4 + 2m^2n^2 + 9n^4 = (m^4 + 6m^2n^2 + 9n^4) - 4m^2n^2 = (m^2 + 3n^2)^2 - 4m^2n^2$$

Recognizing this as a difference of squares, it factors into $(m^2 + 3n^2 + 2mn)(m^2 + 3n^2 - 2mn)$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Line ℓ has a slope of 3 and passes through the point $(0, -1)$. Write the equation of line ℓ in slope-intercept form.

Solution: Slope-intercept form is $y = mx + b$, where m is the slope and b is the y-intercept. We have been given $m = 3$ and $b = -1$, so $\underline{y = 3x - 1}$.

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2. Which of the following cannot be determined from looking at a boxplot (also called a box-and-whisker plot)?

range minimum mean median 3rd quartile

Solution: Given a set of numbers, a boxplot is based on the five-number summary: the minimum and maximum, the quartiles, and the median. The range (maximum minus minimum) can be determined from the length of the "whiskers" of the plot. So only the mean can not be determined from the boxplot.

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3. Simplify: $(c + d)^2 + c^2 - d^2$.

Solution: $(c + d)^2 + c^2 - d^2 = c^2 + 2cd + d^2 + c^2 - d^2 = 2c^2 + 2cd$.

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4. If 7 of them cost \$77, how much will 11 of them cost?

Solution: Each item costs \$11, so 11 would cost \$121.

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5. The perimeter of a square is 20 inches. Find the area of the square (include units).

Solution: The side length of the square is 5 inches, so the area is 25 square inches.

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6. Solve for m: $|m| = 4 - 1$.

Solution: The right side simplifies to 3; $|m|$ equals 3 when $\underline{m = \pm 3}$.

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7. What is the remainder when 2008 is divided by 7?

Solution: $2008 = 7 \times 286 + 6$, so the remainder is $\underline{6}$.

Section B. Each correct answer is worth 2 points.

8. Write the numerical value of $\log_2 9$, rounded to 4 places after the decimal.

Solution 1: Using the change-of-base formula for logarithms: $\log_2 9 = \dfrac{\ln 9}{\ln 2} \doteq 3.1699$.

Solution 2: If $\log_2 9 = x$, then $2^x = 9$. This can be solved graphically on a calculator by finding the intersection of $y = 9$ and $y = 2^x$, or by finding the x-intercept of $y = 2^x - 9$.

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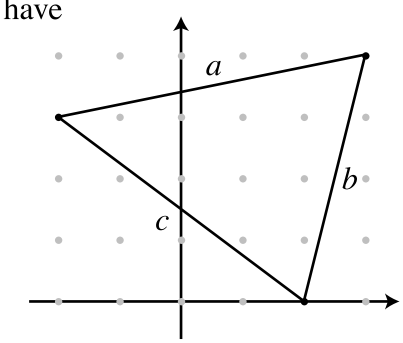
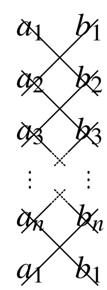
9. Find the area of the triangle whose three vertices are at $(-2, 3)$, $(3, 4)$, and $(2, 0)$.

Solution 1: This triangle fits inside a 4×5 rectangle with vertices $(-2, 0)$, $(-2, 4)$, $(3, 0)$, and $(3, 4)$. The three smaller triangles (outside the desired triangle) have area 2.5, 2, and 6; subtracting these from the area of the rectangle leaves the desired area: $20 - 2.5 - 2 - 6 = 9.5$ square units.



Solution 2: A *lattice point* is a point in the plane with two integral coordinates (such as the vertices of this triangle). According to *Pick's Theorem,* the area of any shape with lattice-point vertices is $A = \frac{1}{2}B + I - 1$, where B is the number of lattice points on the boundary and I is the number of lattice points in the interior. For this triangle, $B = 3$ and $I = 9$, so $A = 1.5 + 9 - 1 = 9.5$.

Solution 3: The *shoelace theorem* says that to find the area contained inside a polygon with vertices (a_1, b_1), (a_2, b_2), ..., (a_n, b_n), listed in clockwise order, you "stack" the coordinates as shown on the right (note that the frst pair of vertices is repeated at the end of the list). Next, draw "laces" between adjacent diagonal pairs of coordinates. Find the sum S_1 of the products from negatively-sloped pairs ($\searrow$), and the sum S_2 of the products of the other ($\swarrow$) pairs. The area is the $A = \frac{1}{2}|S_1 - S_2|$. For this triangle,



$$S_1 = (-2)(4) + (3)(0) + (2)(3) = -2$$
$$S_2 = (3)(3) + (4)(2) + (0)(-2) = 17$$

so $A = \frac{1}{2}|-2 - 17| = \frac{19}{2} = 9.5$.

Solution 4: A fourth approach—not recommended without a calculator—is to use Heron's formula (also called Hero's formula), which says that the area of a triangle with side lengths a, b, c and semiperimeter $s = \frac{1}{2}(a + b + c)$ is $\sqrt{s(s-a)(s-b)(s-c)}$. Using the distance formula, the three side lengths are $a = \sqrt{26}$, $b = \sqrt{17}$, and $c = \sqrt{25} = 5$. The semiperimeter is therefore $s \doteq 7.111$, and the area is $\sqrt{90.25} = 9.5$ square units.

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10. In the expansion of the binomial $(x - 2y)^{10}$, find the term that includes x^8.

Solution: $(x - 2y)^{10} = x^{10} + \binom{10}{1}x^9(-2y)^1 + \binom{10}{2}x^8(-2y)^2 + \cdots$. Since $\binom{10}{2} = 45$, the desired term is $180x^8y^2$.

11. A rhombus is inscribed in an ellipse, and its four vertices are on the x- and y-axes. The equation of the ellipse is $25x^2 + 16y^2 = 400$. Find the area of the rhombus.

Solution 1: The ellipse intersects the axes at $(\pm 4, 0)$ and $(0, \pm 5)$; these are the vertices of the rhombus. The four quadrants divide the rhombus into four congruent triangles, each with area $\frac{1}{2}(4)(5) = 10$, so the area of the rhombus is 40 square units.

Solution 2: The area of a rhombus is $\frac{1}{2}d_1 d_2$, where d_1 and d_2 are the lengths of the diagonals. From the observations made in Solution 1, the diagonals have length 10 and 8, so the area is $\frac{1}{2}(10)(8) = 40$.

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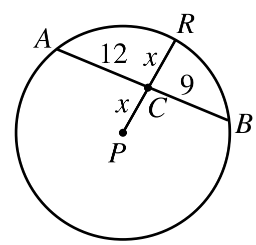
12. Express 111_{five} as a number in base 4.

Solution: $111_{\text{five}} = 1 \times 5^2 + 1 \times 5 + 1 = 31_{\text{ten}}$. Written in terms of powers of 4, $31 = 1 \times 4^2 + 3 \times 4 + 3 = 133_{\text{four}}$.

Section C. Each correct answer is worth 3 points.

13. Given $\odot P$ as shown, with chord $\overline{AB}$ and radius $\overline{PR}$, find the length of the diameter of $\odot P$.



Solution 1: Extend $\overline{PR}$ to a diameter, intersecting $\odot P$ a second time at S. The *intersecting chord theorem* says that, for chords $\overline{AB}$ and $\overline{RS}$, we have $(AC)(CB) = (RC)(CS)$. Therefore, $(12)(9) = (x)(3x)$, so $3x^2 = 108$, or $x = 6$. The diameter is therefore $4x = 24$.

Solution 2: Draw radius $\overline{PQ}$ perpendicular to chord $\overline{AB}$; let D be the intersection of $\overline{PQ}$ and $\overline{AB}$. Then D must be the midpoint of $\overline{AB}$. Let $t = PD$, and note that $PB = 2x$ and $PC = 1.5$. For right $\triangle PDC$ and right $\triangle PDB$, the Pythagorean theorem gives (respectively) $t^2 + 1.5^2 = x^2$ and $t^2 + 10.5^2 = (2x)^2$. Solving both equations for t^2 leads to $x^2 - 2.25 = 4x^2 - 110.25$, so $3x^2 = 108$. Proceed as in Solution 1.

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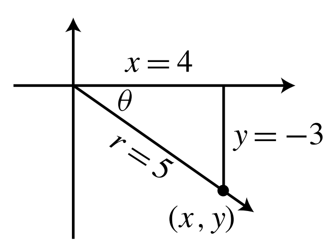
14. If $\cos(\theta) = 0.8$ and $\tan(\theta) = -0.75$, find the exact value of $\sin(\theta)$.

Solution 1: Because $\tan(\theta) = -0.75 = \dfrac{\sin(\theta)}{\cos(\theta)}$, we know that $\sin(\theta) = -0.75 \times 0.8 = -0.6$.

Solution 2: The terminal ray of θ must be in quadrant IV, because the cosine is positive in quadrants I and IV, while the tangent is negative in quadrants II and IV. Choose the point (x, y) on the terminal ray so that r, the distance from the origin to (x, y), equals 5. Since $\cos\theta = \frac{4}{5} = \frac{x}{r}$, we have $x = 4$. Then the triangle shown in the figure is a 3-4-5 right triangle, which means $y = -3$. Therefore, $\sin\theta = \frac{y}{r} = \frac{-3}{5} = -0.6$.



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15. A door 2 ft wide and 7 ft high is swung through a 60° arc. Find the volume of the region swept out by the door. Express your answer to the nearest hundredth cubic feet.

Solution: The volume of a right circular cylinder is $\pi r^2 h$. The volume swept out by the door is $\frac{1}{6}$ of a cylinder with radius $r = 2$ ft and height $h = 7$ ft, so the volume is $\frac{14}{3}\pi \doteq 14.66$ cubic feet.