BLUFFTON

EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2007
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. List all prime factors of 12.

Solution: The prime factorization of 12 is $2 \cdot 2 \cdot 3$, so the prime factors are 2 and 3.

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2. (a) Find the name (capital letter) of the midpoint of $\overline{AS}$.
   (b) If F is the midpoint of $\overline{AX}$, give the coordinate of point X.

Solution: (a) H is halfway between A and S. (Its coordinate is the average of the coordinates of A and S.) (b) The coordinate of F (2) must be the average of the coordinates of A and X, so $2 = \frac{-3+x}{2}$. Therefore, $x = 7$.

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3. Along with the traditional Pi (Π) Day on March 14, some mathematicians like to have a second celebration, called "Fall Π Day." If Fall Π Day is considered to be the 314th day of the year, what is the date for Fall Π Day this year (2007)?

Solution: 314 days is 10 full months and a little bit more. Adding up the days in each month from January through October gives

$$31 + 28 + 31 + 30 + 31 + 30 + 31 + 31 + 30 + 31 = 304,$$

so Fall Π Day is November 10.

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4. Express $\dfrac{2007}{7002}$ as a fraction in simplest form.

Solution: The greatest common divisor of 2007 and 7002 is 9, so $\frac{2007}{7002} = \frac{2007/9}{7002/9} = \frac{223}{778}$. (This can also be determined by trial and error, by dividing numerator and denominator by 3 twice, and then confirming that there are no other common factors.)

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5. Solve for all real values of x: $x^2 = 4x$.

Solution: $x^2 = 4x$ when $x^2 - 4x = 0$, or $x(x-4) = 0$, so either $x = 0$ or $x = 4$.

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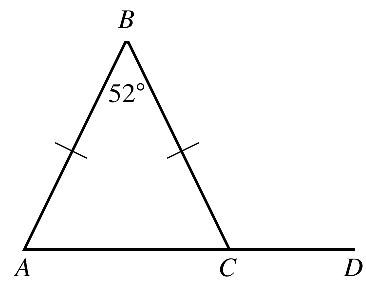
6. Express as a decimal: $5^{-2} + 10^{-1}$.

Solution: $5^{-2} + 10^{-1} = \frac{1}{25} + \frac{1}{10} = 0.04 + 0.1 = 0.14$.

7. Given the figure as shown, find $m\angle BCD$.



Solution: Since $\triangle ABC$ is isosceles, the base angles $\angle BAC$ and $\angle BCA$ are congruent. Since the internal angles of any triangle must add to $180°$, those two angles have measure $64°$, so $m\angle BCD = 116°$.

Section B. Each correct answer is worth 2 points.

8. For the seven numbers $\{ x, 17, x+4, 4x-3, -16, 9, x-4 \}$, the mean is 4. What is the median of these seven numbers?

Solution: The sum of the seven numbers is $7x+7$. The average is therefore $\frac{7x+7}{7} = x+1 = 4$, so $x = 3$. Knowing this, the list of numbers is $\{ 3, 17, 7, 9, -16, 9, -1 \}$. The median of this list—that is, the number in the middle when then numbers are put in order—is 7.

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9. Simplify: $\dfrac{a^3-b^3}{a^2-b^2}$.

Solution: The numerator (a difference of cubes) factors into $(a-b)(a^2+ab+b^2)$. The denominator (a difference of squares) factors into $(a-b)(a+b)$. Therefore,

$$\frac{a^3-b^3}{a^2-b^2} = \frac{a^2+ab+b^2}{a+b}.$$

This can also be written as $\dfrac{a^2+2ab+b^2-ab}{a+b} = \dfrac{(a+b)^2-ab}{a+b} - a + b - \dfrac{ab}{a+b}$.

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10. Given that $4^{20}+4^{20} = 2^x$, find x.

Solution: $4^{20}+4^{20} = 2\cdot 4^{20} = 2\cdot 2^{40} = 2^{41}$, so $x = 41$.

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11. Evaluate: $\left(\begin{vmatrix} 10 & 7 \\ 6 & 5 \end{vmatrix}\right)^2 - \sum_{k=1}^{4} k$. (*Note:* $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is the determinant of that matrix.)

Solution: The determinant $\begin{vmatrix} 10 & 7 \\ 6 & 5 \end{vmatrix} = 10\cdot 5 - 6\cdot 7 = 8$, and $\sum_{k=1}^{4} k = 1+2+3+4 = 10$ (the 4th triangular number), so $\left(\begin{vmatrix} 10 & 7 \\ 6 & 5 \end{vmatrix}\right)^2 - \sum_{k=1}^{4} k = 8^2 - 10 = 54$.

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12. Find the center and the radius of the circle $3x^2+3y^2-24y+18x = 25$.

Solution: Begin by completing the squares on the left side:

$$3x^2+3y^2-24y+18x = 3(x^2+6x)+3(y^2-8y) = 3(x+3)^2+3(y-4)^2-27-48.$$

Substituting this in the original equation gives $3(x+3)^2+3(y-4)^2-27-48 = 25$, so $3(x+3)^2+3(y-4)^2 = 100$, or $(x+3)^2+(y-4)^2 = \frac{100}{3}$. Therefore, the center is $(-3, 4)$ and the radius is $\sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$.

Section C. Each correct answer is worth 3 points.

13. Solve the system on the right and express your answer as an ordered pair (x, y). Note that k is a constant real number.

Solution: For the system $\begin{cases} 3x + ky = 6 \\ 2x - 3y = 6 \end{cases}$, the second equation leads to $x = 1.5y + 3$. Substituting into the first equation give $4.5y + 9 + ky = 6$, so $(4.5 + k)y = -3$, or $y = \frac{-3}{k+0.45} = \frac{-6}{2k+9}$. Substituting this back into $x = 1.5y + 3$ gives $x = \frac{-9}{2k+9} + 3 = \frac{6k+18}{2k+9}$. Therefore, the ordered-pair solution is

$$\left(\frac{6k+18}{2k+9}, \frac{-6}{2k+9}\right).$$

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14. Solve for all values of α $(0^\circ \le \alpha \le 180^\circ)$ such that $\sin^2\alpha - \cos\alpha - \cos^2\alpha = 1$.

Solution: Using the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$, this equation becomes $(1 - \cos^2\alpha) - \cos\alpha - \cos^2\alpha = 1$, or $2\cos^2\alpha + \cos\alpha = 0$. Factoring leads to $(\cos\alpha)(2\cos\alpha + 1) = 0$, which happens when $\cos\alpha = 0$ or $\cos\alpha = -\frac{1}{2}$. The former implies $\alpha = 90^\circ$; the latter implies $\alpha = 120^\circ$.

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15. In the expansion of $(a + b)^n$, there are $n + 1$ individual terms. Find the number of individual terms in the expansion of $(a + b + c)^{10}$.

Solution 1: Try a simpler question first, and try to see a pattern: $(a + b + c)^0 = 1$ has 1 term. $(a + b + c)^1 = a + b + c$ has 3 terms. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ has 6 terms. The expansion of $(a + b + c)^4$ would have 10 terms. This sequence $(1, 2, 6, 10, \dots)$ is the triangular numbers, so for $(a + b + c)^{10}$, the number of terms would be the 11th triangular number: $1 + 2 + 3 + 4 + \cdots + 10 + 11 = 66$.

Solution 2: Each term will be of the form $Ka^r b^s c^t$, where r, s, t are nonnegative integers with $r + s + t = 10$. From this viewpoint, this question is asking, "How many such sets of integers are there?" One way to answer this uses the "stars and bars" method (do a Web search for "stars and bars counting" to learn more). In this case, we consider a set of twelve blanks, in which we will write 10 "stars" and 2 "bars." Then we let r be the number of stars to the left of the first bar, s be the number of stars between the two bars, and t be the number of stars to the right of the second bar. Here are two such configurations:

$$\underline{\star}\ \underline{\perp}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\perp}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star} \qquad \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\perp}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\star}\ \underline{\perp}$$

The first configuration leads to $r = 1, s = 5, t = 4$, while the second gives $r = 4, s = 6, t = 0$. Following this rule, r, s, t will always be three nonnegative integers adding to 10. There are "12 choose 2" $= \binom{12}{2} = 66$ such arrangements of stars and bars, so there are 66 such sets of integers.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Which of these numbers has the largest prime factor? $39, 40, 51, 77, 91, 108, 121$

Solution: Factoring each number gives the answer: $39 = 3 \cdot 13$, $40 = 2^3 \cdot 5$, $51 = 3 \cdot 17$, $77 = 7 \cdot 11$, $91 = 7 \cdot 13$, $108 = 2^2 \cdot 3^3$, $121 = 11^2$. The largest prime factor (17) occurs in 51.

2. Find the product of the squares of the first five natural numbers.

Solution: $1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 = 14,400$. Also accepted: 0.

Note: The standard definition (that is, the definition found in most, if not all, high school math textbooks) of natural numbers is the positive integers $\{1, 2, 3, \ldots\}$. However, the term sometimes refers to the nonnegative integers $\{0, 1, 2, \ldots\}$. With that definition, the answer is 0.

Three contestants answered 0 for this question; this had originally been counted wrong, but those students had one point added to the score later. (This had no effect on prizes.) For more, see `wikipedia.org/wiki/Natural_number` or `mathworld.wolfram.com/NaturalNumber.html`.

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3. The sum of eleven consecutive odd numbers is 2277. Find the largest of these eleven numbers.

Solution: The average of those 11 numbers is $\frac{2277}{11} = 207$, which must be the 6th of the 11 consecutive odd numbers. Therefore, the largest number is 217.

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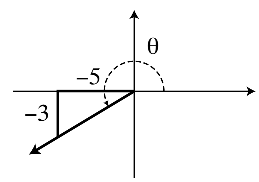
4. Given that $\tan\theta = \frac{3}{5}$, with the terminal side of θ in Quadrant III, find the exact value of $\sin\theta$.

Solution: The diagram on the right shows θ, with the terminal side shown as the hypotenuse of a right triangle. Since $\tan\theta = \frac{3}{5} = \frac{y}{x}$, the legs of the triangle have length 5 and 3; since the terminal side is in Quadrant III, both x and y are negative. Then $\sin\theta = \frac{y}{r}$, where r is the length of the hypotenuse: $r^2 = 3^2 + 5^2$, so $\sin\theta = -\frac{3}{\sqrt{34}}$.



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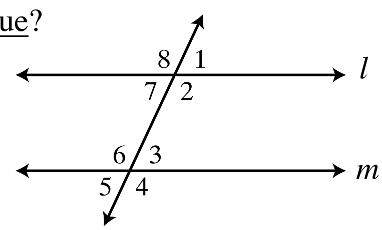
5. In the figure, lines l and m are parallel. Which statements are true?

I. $\angle 1 \cong \angle 5$ II. $\angle 8 \cong \angle 6$ III. $\angle 4 \cong \angle 7$

(A) I only **(B)** II only **(C)** III only

(D) I and II only **(E)** I, II, and III



Solution: When two parallel lines (like l and m) are cut by a transversal (the third line in the diagram), pairs of corresponding angles are congruent; in particular, that means $\angle 8 \cong \angle 6$, so II is true. Also, pairs of alternating exterior angles are congruent, so $\angle 1 \cong \angle 5$; that is, I is true. As for III, $\angle 4$ and $\angle 7$ are supplementary, but are not congruent (unless the transversal is perpendicular to l and m). Therefore, the answer is D.

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6. Solve for all integral values of x: $-4 \le |x| < 2$.

Solution: The only integers with absolute value between −4 and 2 are −1, 0, and 1.

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7. Give an example of a rectangle whose perimeter (in inches) has the same numerical value as its area (in square inches).

Solution: The simplest answer is a 4×4 square. (Not surprisingly, this was also the most common answer.)

More generally, we need L and W which satisfy $LW = 2L + 2W$, or $W = \frac{2L}{L-2}$, so any choice of $L > 2$ yields a corresponding value of W. (For example, $L = 3$ and $W = 6$, or $L = 10$ and $W = 2.5$, or $L = 2.01$ and $W = 402$.)

Section B. Each correct answer is worth 2 points.

8. In the sequence $\ldots, a, b, c, d, e, 0, 1, 1, 2, 3, 5, 8, \ldots$, each term is the sum of the two terms immediately to its left. Find a.

Solution: Work backwards from e to a: $e + 0 = 1$, so $e = 1$. $d + e = 0$, so $d = -1$. $c + d = e$, so $c = 2$. $b + c = d$, so $b = -3$. Finally, $a + b = c$, so $\underline{a = 5}$.

Note: The terms starting with $1, 1, 2, \ldots$ are the familiar *Fibonacci sequence*. The numbers $e, d, c, b, a, \ldots$ are the Fibonacci sequence with alternating signs: $1, -1, 2, -3, 5, \ldots$.

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9. If Δ represents an operation defined as $m \Delta n = m^n - m$, find $2\Delta(3\Delta 2)$.

Solution: $3\Delta 2 = 3^2 - 3 = 6$, so $2\Delta(3\Delta 2) = 2\Delta 6 = 2^6 - 2 = 64 - 2 = \underline{62}$.

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10. Given $f(x) = x^3 + x^2 - 4x - 4$, find the sum of all the intercepts when $f(x)$ is graphed on the x-y coordinate system.

Solution 1: Graph the function on your calculator and observe the roots (zeros) at $x = -2, x = -1$ and $x = 2$, and the y-intercept at $y = -4$. The sum is $\underline{-5}$.

Solution 2: The x-intercepts are the solutions to $f(x) = 0$. In factored form, $f(x) = (x+1)(x+2)(x-2)$, so the three x-intercepts are -1, -2, and 2. Add to this the y-intercept, which is $f(0) = -4$; the sum is $\underline{-5}$.

Note: Because some students may have been taught to think of intercept as an ordered pair (x, y), we were willing to accept $(-2, 0) + (-1, 0) + (2, 0) + (0, -4) = \underline{(-1, -4)}$—even though "adding" ordered pairs is a strange concept.

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11. Note that on this part of the contest, there are 2 ways to score 2 points (either get 2 correct from Section A, or 1 correct from Section B). Similarly, there are 3 ways to score 3 points ($1+1+1$, or $1+2$, or 3), and 4 ways to score 4 points ($1+1+1+1$, or $1+1+2$, or $1+3$, or $2+2$). How many different ways can you score 11 points on this part? (It is not 11.)

Solution: Grouping by the number of correct 3-pointers, we can make a list of every possible combination of points that totals 11:

Number of 3-pointers	Additional points needed	Ways to score	
3	2	2	$1+1$, or 2
2	5	3	1×5, $1 \times 3 + 2$, or $1 + 2 + 2$
1	8	4	$1 \times 6 + 2$, $1 \times 4 + 2 + 2$, $1 + 1 + 2 \times 3$, 2×4
0	11	4	$1 \times 7 + 2 + 2$, $1 \times 5 + 2 \times 3$, $1 \times 3 + 2 \times 4$, $2 + 2 \times 5$

Note: If we had an unlimited number of 1, 2, and 3 point questions, we could generalize this question to the number of ways to score n points in basketball (by a combination of 1-, 2-, and 3-pointers). The general answer to that question is $\left\lfloor \frac{1}{12}n^2 + \frac{1}{2}n + 1 \right\rfloor$, where $\lfloor x \rfloor$ is the "floor" (greatest integer) function. For example, there are $\left\lfloor 16\frac{7}{12} \right\rfloor = 16$ ways to score $n = 11$ points in basketball (the 13 listed above, plus 1×11, $1 \times 9 + 2$, and $1 \times 8 + 3$).

12. Sarah drove 30 miles from home to the Bluffton University Mathematics Contest in 50 minutes. How fast (in mph) must she drive home in order to average 40 mph for the entire trip?

Solution: To average 40 mph for a 60-mile trip, she must complete the trip in a total time of 1.5 hours (since $\frac{60 \text{ miles}}{1.5 \text{ hours}} = 40$ mph). Therefore, she must return home in 40 minutes, meaning her average speed going home is $\frac{30 \text{ miles}}{2/3 \text{ hour}} = \underline{45 \text{ mph}}$.

Section C. Each correct answer is worth 3 points.

13. A set of N numbers has sum S. Each number in the set is increased by 20, then multiplied by 4, then decreased by 20. Find the sum of the numbers in the new set (in terms of N and S).

Solution: If each number is increased by 20, the sum is increased by $20N$; that is, $S^* = S + 20N$. Then, when each number is multiplied by 4, S^* is also multiplied by 4: $S^{**} = 4S^*$. Finally, we decrease the sum by $20N$: $S^{***} = S^{**} - 20N$. Therefore, the final sum is

$$S^{***} = S^{**} - 20N = 4S^* - 20N = 4(S + 20N) - 20N = 4S + 60N.$$

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14. The vertices of $\triangle ABC$ are $A(0, 0)$, $B(5, -2)$, and $C(1, -4)$. Find the measure of the smallest angle, rounded to the nearest degree.

Solution: The lengths of the three sides are $AB = \sqrt{5^2 + 2^2} = \sqrt{29}$, $AC = \sqrt{1^2 + 4^2} = \sqrt{17}$, and $BC = \sqrt{4^2 + 2^2} = \sqrt{20}$. The smallest angle is opposite the shortest side: $\angle ABC$. Let $\theta = m\angle ABC$; then (using the law of cosines) $AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos\theta$, or

$$17 = 29 + 20 - 2\sqrt{29 \cdot 20}\cos\theta.$$

Therefore, $\cos\theta = 0.66436\ldots$, so $\theta = \cos^{-1}(0.66436\ldots) = 48.366\ldots°$. Rounded to the nearest degree, the answer is $\underline{48°}$.

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15. A single die (number cube) is tossed 3 times. What is the probability that exactly 2 of the numbers will be the same?

Solution: There are $6^3 = 216$ possible sequences of rolls (all equally likely). In order for exactly two numbers to be the same, the die must fall in one of the patterns aab, aba, or baa. There are 6 possible choices for a (the pair), and 5 choices for b, so altogether there are $3 \times 6 \times 5 = 90$ such sequences of rolls. Therefore, the probability of having exactly two the same is $\frac{90}{216} = \frac{5}{12}$.