BLUFFTON
EMBRACING SPIRIT, ENGAGING MINDS

MATHEMATICS CONTEST 2006
SOLUTIONS

PART I: 30 Minutes; NO CALCULATORS

<u>Section A</u>. Each correct answer is worth 1 point.

1. Evaluate: $2^0 + 0^6$

<u>Solution:</u> $a^0 = 1$ for any $a \neq 0$, and $0^b = 0$ for any $b \neq 0$. Therefore, $2^0 + 0^6 = 1 + 0 = 1$.

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2. The perimeter of a rectangle is 2006 inches. Its length is 800 inches. What is its <u>width</u>?

<u>Solution:</u> Perimeter equals $2L + 2W$. If $L = 800$, $2W = 406$, so $W = 203$ inches.

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3. Express in simplest form: $\frac{1}{10} - \frac{1}{15} + \frac{1}{20} - \frac{1}{30} + \frac{1}{40}$

<u>Solution:</u> The common denominator for these fractions is 120, so

$$\frac{1}{10} - \frac{1}{15} + \frac{1}{20} - \frac{1}{30} + \frac{1}{40} = \frac{12-8+6-4+3}{120} = \frac{9}{120} = \frac{3}{40}.$$

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4. Evaluate and write the answer in scientific notation: $(280 \times 10^6) \div (7 \times 10^3)$

<u>Solution:</u> $\dfrac{280 \times 10^6}{7 \times 10^3} = 40 \times 10^{6-3} = 40 \times 10^3 = 4 \times 10^4$

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5. <u>How many</u> of these numbers are <u>not</u> divisible by 9? 27, 90, 108, 792, 162, 2006, 5445

<u>Solution:</u> The "brute force" approach would be to divide each of these numbers by 9, and see which have a remainder. There is a faster approach of you know that a number is divisible by 9 if (and only if) the sum of its digits is divisible by 9. For example, 108 is divisible by 9 because $1 + 0 + 8 = 9$ is (obviously) divisible by 9. By this test, only 2006 is not divisible by 9, so the answer is <u>1</u>.

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6. Evaluate: $12 + 3 \times 4 - 6 + 8$

<u>Solution:</u> Follow standard order of operations: $12 + 3 \times 4 - 6 + 8 = 12 + 12 - 6 + 8 = 26$.

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7. The divisor is 11, the quotient is 3, and the remainder is 4. What is the dividend?

<u>Solution:</u> In the equation $a \div b = c + \frac{d}{b}$, often written as $a = bc + d$, the dividend is a, the divisor b, the quotient c, and the remainder d. Therefore, the dividend is $a = (11)(3) + 4 = 37$.

Section B. Each correct answer is worth 2 points.

8. Which one of the following expressions is not equal to the other four?
   (A) $3x(x+2)(x-1)$ **(B)** $-(1-x)(3x^2+6x)$ **(C)** $3(x^3+x^2-2)$
   (D) $(x+2)(3x^2-3x)$ **(E)** $-6x+3x^2+3x^3$

Solution: C is different from the rest. This observation can be made in a number of ways; for example, by noting that C does not have a common factor of x (that is, all the other expressions equal 0 when $x=0$).

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9. Express as a fraction in simplest form: $0.1\overline{23}$

Solution 1: If $x = 0.1\overline{23}$, then $100x = 12.3\overline{23}$, so $100x - x = 99x = 12.2$. Therefore, $x = \frac{12.2}{99} = \frac{122}{990} = \frac{61}{495}$.

Solution 2: Note that $0.1\overline{23} = 0.\overline{32} - 0.2 = \frac{32}{99} - \frac{1}{5} = \frac{32 \cdot 5 - 99}{99 \cdot 5} = \frac{160-99}{495} = \frac{61}{495}$.

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10. Factor completely: $ax - by - bx + ay$

Solution: Group like terms:

$$ax - by - bx + ay = ax + ay - by - bx = a(x+y) - b(x+y) = (a-b)(x+y).$$

Equivalently,

$$ax - by - bx + ay = ax - bx - by + ay = x(a-b) + y(-b+a) = (a-b)(x+y).$$

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11. The degree measure of an angle of a regular polygon cannot be
    (A) 60 **(B)** 75 **(C)** 90 **(D)** 120 **(E)** 135

Solution: The answer is B: The total degree measure of an n-gon is $180(n-2)$. (For example, the angles in a triangle sum to $180°$; in a quadrilateral, they sum to $360°$, etc.) Therefore, in a regular n-gon, an individual angle measure is $\frac{180(n-2)}{n}$. This gives $60°$ when $n = 6$ (a regular hexagon), $90°$ for $n = 4$ (a square), $120°$ for $n = 3$ (an equilateral triangle), and $135°$ for $n = 8$ (a regular octagon).

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12. Write in simplest form as an exponential expression with a single base: $3^{3x} \cdot 3^{2(x+1)}$

Solution: Use the rule of exponents that says $a^b \cdot a^c = a^{b+c}$:

$$3^{3x} \cdot 3^{2(x+1)} = 3^{3x+2(x+1)} = 3^{3x+2x+2} = 3^{5x+2}.$$

Section C. Each correct answer is worth 3 points.

13. If (x^3+3x-2) is divided by (x^2-2), what is the remainder?

Solution 1: Carry out polynomial long division.

Solution 2: Any time you perform the division $a \div b$ and get quotient q and remainder r, those four quantities are related by the equation $a = b \cdot q + r$. (See the solution to problem #7 on the previous page.) For polynomial division, the remainder r should be a polynomial of degree smaller than the degree of b. So we want r (a first-degree polynomial) in the equation

$x^3 + 3x - 2 = (x^2 - 2)q + r$. In order to have an x^3 term (but no x^2 term) on the right side, q must be x, so $x^3 + 3x - 2 = (x^2 - 2)(x) + r$, or $r = (x^3 + 3x - 2) - (x^2 - 2x) = 5x - 2$.

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14. Express as a single radical in simplest form: $(\sqrt{2})(\sqrt[3]{5})$

Solution: $(\sqrt{2})(\sqrt[3]{5}) = 2^{1/2} \cdot 5^{1/3} = 2^{3/6} \cdot 5^{2/6} = (2^3 \cdot 5^2)^{1/6} = 200^{1/6} = \sqrt[6]{200}$.

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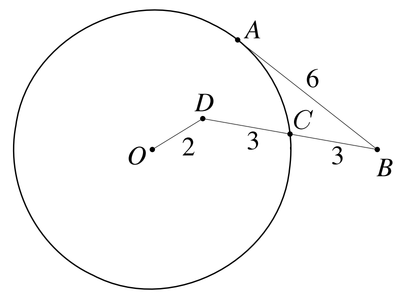
15. (See figure). $\overline{AB}$ is tangent to $\odot O$ at A. If $BC = 3, CD = 3, OD = 2$, and $AB = 6$, find the length of the radius of $\odot O$.

Solution: Extend segment $\overline{BD}$ to intersect the circle at E. For any tangent segment (like $\overline{AB}$) and secant segment (like $\overline{BE}$), the lengths are related by the equation $AB^2 = (BC)(BE)$. Therefore, $BE = 12$, and $DE = 6$.



Method 1: Extend $\overline{OD}$ into a diameter, intersecting the circle at points F and G. A formula similar to the one used above says that for any two secants (like $\overline{FG}$ and $\overline{CE}$) which intersect inside the circle, we have $(CD)(DE) = (FD)(DG)$. Therefore, $(3)(6) = (r-2)(r+2)$, so $r^2 = 22$, and $r = \sqrt{22}$.

Method 2: Let H be the midpoint of $\overline{EC}$, and note that $\overline{OH} \perp \overline{EC}$. Let $x = OH$. Then by the Pythagorean theorem (applied to $\triangle OHD$), $x^2 + 1.5^2 = 2^2$, so $x^2 = 1.75$. By the Pythagorean theorem (applied to $\triangle OHC$), $x^2 + 4.5^2 = r^2$, so $r^2 = 22$, and $r = \sqrt{22}$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. Evaluate: $4^5 - 5^4$

Solution: This is simple on a calculator, but even without one, we can compute $4^5 = 4^2 \cdot 4^2 \cdot 4 = 16^2 \cdot 4 = 256 \cdot 4 = 1024$, and $5^4 = 25^2 = 625$. Therefore, $4^5 - 5^4 = 399$.

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2. The area of a triangle is 2006 sq. cm. Its base is 2006 cm. What is its height?

Solution: Area equals $\frac{1}{2}bh = \frac{1}{2}(2006)(h) = 2006$, so $h = 2$ cm.

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3. If $\frac{2}{5}$ of a number is 2.5, what is the number?

Solution: $\frac{2}{5}x = 2.5$, so $x = \frac{2.5}{2/5} = \frac{2.5}{0.4} = 6.25$.

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4. Find the sum of x and y if $3x = 90$, and $3y = x$.

Solution: Solve $3x = 90$ to find $x = 30$, then solve $3y = x$ to find $y = 10$. Therefore, $x + y = 30 + 10 = 40$.

5. Find the measure (to the nearest degree) of the acute angle whose tangent is 1.732050808.

<u>Solution 1:</u> Put your calculator in degree mode, and compute $\tan^{-1}(1.732050808)$, which should give $60°$(or 60.0000000062, but the problem asks for the answer to the nearest degree).

<u>Solution 2:</u> If you recognize that $1.732050808 \doteq \sqrt{3}$, you can draw a right triangle whose legs have length 1 and $\sqrt{3}$, so that the tangent of the larger acute angle would be $\sqrt{3}$. The hypotenuse length is therefore 2; these are the proportions for a 30–60–90 right triangle, meaning that the larger acute angle is a $60°$ angle.

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6. The average of 11 numbers is 121. When one number is dropped, the average of the remaining set of numbers is 120. What number was dropped?

<u>Solution:</u> If the average of 11 numbers is 121, their sum must be $(11)(121) = 1331$. If 10 numbers have average 120, their sum must be $(10)(120) = 1200$. Therefore, the missing number is 131.

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7. Give an example of two numbers such that their <u>product</u> is <u>positive</u>, and their <u>sum</u> is <u>negative</u>.

<u>Solution:</u> This is two for any two negative numbers, for example: $(-1)(-2) = 2$ is positive, but $(-1) + (-2) = -3$ is negative.

<u>Section B</u>. Each correct answer is worth 2 points.

8. Take a three-digit number ending in 1, say $ab1$. The sum of the digits, $a + b + 1$, is a <u>two</u>-digit number, cd. The product of those digits, $c \cdot d$, equals 8. List all possible values for the original number.

<u>Solution:</u> If $c \cdot d = 8$, then the two-digit number cd must be one of these: 18, 81, 24, 42. But $a \leq 9$ and $b \leq 9$, so $a + b + 1$ cannot add up to more than 19. Therefore, $a + b + 1 = 18$, which means a and b are the digits 8 and 9 (in either order); the original number was therefore 891 or 981.

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9. A single die (number cube) is tossed three times. What is the probability that all three numbers are the same? Express as a ratio in simplest form.

<u>Solution 1:</u> Here is one line of reasoning: There are $6 \cdot 6 \cdot 6 = 6^3 = 1296$ possible outcomes from rolling the die three times. Of those 1296 possibilities, 6 have all numbers the same—either 3 ⚀s, 3 ⚁s, . . . , or 3 ⚅s. Therefore, the probability that all faces are the same is $\frac{6}{1296} = \frac{1}{36}$.

<u>Solution 2:</u> Roll the die once. Regardless of how it turns up, there is a $\frac{1}{6}$ chance that the second roll is the same as the first roll, and a $\frac{1}{6}$ chance that the third roll is the same. Therefore, the probability that both rolls are the same as the first is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

10. Doc Math chose a two-digit number. He subtracted it from 300 and doubled the result. Then he added the original two-digit number to this result. What is the largest number that Doc Math could get?

Solution: If x is the two-digit number, Doc Math has computed $(300 - x) \cdot 2 + x = 600 - x$. Because $x \geq 10$, his final result is $600 - x \geq 590$.

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11. While driving, Dale Jr. noticed that his car's odometer reading, 47974 miles, was a palindrome (reads the same forward as backward). Dale Jr. continued driving, and two hours later the odometer showed the next possible palindrome. What was the average speed of his car during those 2 hours (in mph)?

Solution: The next palindrome after 47974 is 48084, 110 miles later. Therefore, Dale Jr. traveled 110 miles in 2 hours, for an average speed of $\frac{110}{2} = 55$ mph.

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12. Find the largest integer n for which 10^n is a factor of $(8^4)(200^6)(15^{13})$.

Solution: We have $(8^4)(200^6)(15^{13}) = (2^{12})(2^6 \cdot 10^{12})(5^{13} \cdot 3^{13}) = 2^{18} \cdot 10^{12} \cdot 5^{13} \cdot 3^{13}$. That gives 12 "obvious" factors of 10, and 13 additional factors of 10 from combining paired factors of 2 and 5, for a total of $n = 25$ factors of 10.

Section C. Each correct answer is worth 3 points.

13. In a set of ten whole numbers, the minimum number is 62, the range is 25, and the median is 82. Find the smallest and largest possible values for the mean of that set of numbers (given to the nearest tenth).

Solution: The answer is 74.5 to 82. The smallest number is 62, the largest is $62 + 25 = 87$, and the middle two numbers average to 82. That is, if listed in order, the 10 numbers are

$$62, \_\_, \_\_, \_\_, \underline{a}, \underline{b}, \_\_, \_\_, \_\_, 87$$

where $\frac{a+b}{2} = 82$. The smallest (or largest) possible means happen when the blanks have the smallest (or largest) possible numbers in them:

62, 62, 62, 62, 82, 82, 82, 82, 82, 87, for which the mean is 74.5, and
62, 82, 82, 82, 82, 82, 87, 87, 87, 87, for which the mean is 82.

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14. Given a triangle with sides 6, 7, and 9 units, write the length of the altitude to the shortest side. The answer may be exact, or to the nearest hundredth.

Solution: Given the three side lengths a, b, c of a triangle, we can find its area using Heron's formula: compute the semiperimeter $s = (a+b+c)/2$, then find $A = \sqrt{s(s-a)(s-b)(s-c)}$. For this triangle, $s = 11$, so the area is $A = \sqrt{(11)(5)(4)(2)} = \sqrt{440} = 2\sqrt{110}$. Using the more common area formula, with h as the altitude to the shortest side, we have $A = \frac{1}{2}6h = 3h$. Therefore, $h = \frac{2}{3}\sqrt{110} \doteq 6.99$.

15. $\log_{16}(\log_2(\log_3 x)) = \frac{1}{4}$. Write the value of x.

Solution: In general, if $\log_a b = c$, then $a^c = b$. Therefore, $\log_{16}(\log_2(\log_3 x)) = \frac{1}{4}$ means that $\log_2(\log_3 x) = 16^{1/4} = 2$, which in turn means that $\log_3 x = 2^2 = 4$, so $x = 3^4 = 81$.