

PART I: 30 Minutes; NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. List three natural numbers.

Solution: “Natural numbers” means non-negative integers. The most common answer here was “1,2,3.”

2. Two lines that do not lie in the same plane are said to be ? .

Solution: “Skew” or “noncoplanar” were the two accepted answers. (We also learned during the grading process that another possible answer is “agonic.”)

3. Solve for all real numbers: $x^3 = 9x$.

Solution: Rewrite as $x^3 - 9x = 0$, which factors to give $x(x^2 - 9) = 0$, or $(x)(x - 3)(x + 3) = 0$. Therefore, $x = 0$ or $x = 3$ or $x = -3$. (Note: This makes use of the “zero factor property,” which in the simplest case says that, if $ab = 0$, then either $a = 0$, or $b = 0$, or both. This fact is also useful for problem 3 in part II.)

4. $\sqrt{150}$ is between what two whole numbers?

Solution: $12^2 = 144$ and $13^2 = 169$, so $\sqrt{150}$ must be between 12 and 13.

5. Express $(5\sqrt{3})^2$ in simplest form.

Solution: $(5\sqrt{3})^2 = 5^2 \cdot \sqrt{3}^2 = 25 \cdot 3 = 75$.

6. Find the midpoint of the segment whose endpoints are (1000, -3000) and (-3004, 2004).

Solution: The coordinates of the midpoint are found by averaging the given coordinates: $\frac{1000-3004}{2} = -1002$ and $\frac{-3000+2004}{2} = -498$, so the midpoint is $(-1002, -498)$.

7. The ratio of rocks to rolls is 2 to 3, and there are 2004 more rolls than rocks. How many rolls are there?

Solution: If x is the number of rocks and y is the number of rolls, then

$$\frac{x}{y} = \frac{2}{3} \quad \text{and} \quad y - x = 2004.$$

Then $x = \frac{2}{3}y$, so $\frac{1}{3}y = 2004$, and therefore $y = 3 \cdot 2004 = 6012$ rolls.

Section B. Each correct answer is worth 2 points.

8. Solve for m in terms of n : $am - bn = cmn$.

Solution: Collect on the left side all terms which include m , and move everything else to the right: $am - cmn = bn$.

Then factor out m on the left: $m(a - cn) = bn$.

Finally, divide both sides by $a - cn$:

$$m = \frac{bn}{a - cn}.$$

9. Simplify completely: $\frac{x^3 - x^2 - 6x}{x^2 + x - 12}$.

Solution: The numerator factors into $(x)(x - 3)(x + 2)$, while the denominator factors into $(x - 3)(x + 4)$. Cancelling common factors leaves

$$\frac{(x)(x + 2)}{x + 4} = \frac{x^2 + 2x}{x + 4}.$$

10. Express as a rational number in simplest form: $3^{-2} - (-2)^{-3}$.

Solution: $3^{-2} - (-2)^{-3} = \frac{1}{3^2} - \frac{1}{(-2)^3} = \frac{1}{9} + \frac{1}{8} = \frac{8}{72} + \frac{9}{72} = \frac{17}{72}$.

11. Solve for x and y ; express as an ordered pair (x, y) : $\begin{cases} x + y = 11 \\ 3x - 2y = 13 \end{cases}$

Solution 1: (Solve one equation and substitute.) The first equation tells us that $y = 11 - x$, so in the second equation we have $3x - 2(11 - x) = 13$, or $5x - 22 = 13$. Then $5x = 35$, so $x = 7$. Finally, $y = 11 - x = 4$. The solution is therefore $(7, 4)$.

Solution 2: (Eliminate a variable—essentially equivalent to the method above.) Multiply the first equation by 2 and add to the second equation: $(2x + 2y) + (3x - 2y) = (22) + (13)$. Therefore, $5x = 35$, so $x = 7$ Finish as in solution 1.

12. Given $f(x) = x^2 - x - 1$ and $g(x) = 2x + 3$. If $f(x) = 1$, find all values of $g(x)$.

Solution: If $f(x) = 1$, then $x^2 - x - 1 = 1$, so $x^2 - x - 2 = 0$. Solve this either using the quadratic formula, or by factoring: $(x - 2)(x + 1) = 0$, so $x = 2$ or $x = -1$. Therefore, $g(x)$ is either $g(2) = 7$ or $g(-1) = 1$.

Section C. Each correct answer is worth 3 points.

13. $2004_{\text{six}} = \underline{\quad?} \text{ five}$. (The subscripts “six” and “five” mean those numbers are written in base 6 and base 5. For example, $23_{\text{six}} = 15_{\text{ten}}$.)

Solution: Just as

$$2004_{\text{ten}} = 2 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 4 \times 10^0,$$

we have

$$2004_{\text{six}} = 2 \times 6^3 + 0 \times 6^2 + 0 \times 6^1 + 4 \times 6^0 = 2(216) + 4 = 436 \text{ (base ten)}.$$

Now, the powers of 5 are $5^0 = 1, 5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625, \dots$. To convert 436 to base five, subtract powers of 5: $436 - 3 \times 5^3 = 61$; $61 - 2 \times 5^2 = 11$; $11 - 2 \times 5^1 = 1$; so $2004_{\text{six}} = 3221_{\text{five}}$.

14. Solve for all values of x : $\sum_{n=0}^4 (n!) - \sum_{n=0}^4 n = x^2 - 25$.

Solution: Recall that $n! = n(n-1)(n-2) \cdots (2)(1)$, and that $0! = 1$. Therefore,

$$\sum_{n=0}^4 (n!) = 0! + 1! + 2! + 3! + 4! = 1 + 1 + 2 + 6 + 24 = 34.$$

Meanwhile,

$$\sum_{n=0}^4 n = 0 + 1 + 2 + 3 + 4 = 10.$$

Then this equation simplifies to $34 - 10 = x^2 - 25$, or $x^2 = 49$. The solutions are therefore $x = \pm 7$.

15. If $\log_2[\log_3(\log_4 b)] = \log_4[\log_3(\log_2 a)] = 0$, find the ratio of a to b .

Express in simplest form.

Solution: Note that for any c , $\log_c x = 0$ implies $x = 1$, and $\log_c y = 1$ implies $y = c$.

Therefore, both $\log_3(\log_4 b)$ and $\log_3(\log_2 a)$ must equal 1, which in turn means that $\log_4 b$ and $\log_2 a$ both equal 3. Then $b = 4^3$ and $a = 2^3$, so $a/b = 8/64 = 1/8$.

PART II: 30 Minutes; CALCULATORS NEEDED

Section A. Each correct answer is worth 1 point.

1. A plane quadrilateral has integral sides of 1, 2, 4, and x . Find all possible values of x .

Solution: In a quadrilateral, the total length of any three sides must exceed the length of the fourth side. (This is the equivalent of the triangle inequality, which says that for a triangle with side lengths a , b and c , we must have $a + b > c$, $a + c > b$ and $b + c > a$.) Therefore:

$$x + 1 + 4 > 2, \quad x + 2 + 4 > 1, \quad x + 1 + 2 > 4, \quad 1 + 2 + 4 > x.$$

The first two inequalities tell us that $x > -3$ and $x > -5$, which we already knew. The last two tell us that $1 < x < 7$. Because x must be an integer, we conclude that x is in the set $\{2, 3, 4, 5, 6\}$.

2. Express in simplest form: $\frac{a^{4n+3}}{a^{3n-2}}$.

Solution: $\frac{a^{4n+3}}{a^{3n-2}} = a^{4n+3} \cdot a^{-3n+2} = a^{4n+3-3n+2} = a^{n+5}$.

3. Write all solutions which are real and rational: $x(x^2 + 4)(x^2 - 36)(x^2 - 5)(x + 0.7) = 0$.

Solution: Using the zero factor property (see the solution to problem 3 in part I), we have $x = 0$, $x^2 + 4 = 0$, $x^2 - 36 = 0$, $x^2 - 5 = 0$, or $x + 0.7 = 0$. The second equation has no real solutions (the solutions are the complex numbers $x = \pm 2i$). The solutions to $x^2 - 5 = 0$ are irrational ($x = \pm\sqrt{5}$). Only the first, third, and fifth equations have rational solutions: $x = 0$, $x = \pm 6$, or $x = -0.7$.

4. Write the converse of “If Browns, then not Bengals.”

Solution: The converse of the statement “If A then B” is “If B then A.” Therefore: “If not Bengals, then Browns.”

5. Give the last name of either (not both) of the two independent inventors of calculus.

Solution: (Isaac) Newton developed the ideas first, but did not publish them until after (Gottfried) Leibniz had done so.

6. The hypotenuse of a right triangle is 58, and one of the legs is 40. Find the length of the shortest side of that right triangle.

Solution: The length of the remaining side is $\sqrt{58^2 - 40^2} = \sqrt{1764} = 42$, so the shortest side is 40 units long.

7. A single die (number cube) is rolled three times. What is the probability that the same number comes up all three times? Express as a fraction in simplest form.

Solution: Regardless of how the first roll turns out, the second roll is the same with probability $1/6$, and the third roll is the same with the same probability. So, all three rolls are the same with probability $(\frac{6}{6})(\frac{1}{6})(\frac{1}{6}) = \frac{1}{36}$.

Section B. Each problem is worth 2 points.

8. The six-digit number $2a00b4$ is divisible by 9 and divisible by 11. What is the number?

Solution: This can be solved by trial and error—but with 100 different combinations of a and b , that could be quite time-consuming! A faster approach is to use the following facts about numbers:

- (1) If a number is divisible by 9, then the sum of its digits is also divisible by 9. (For example, $5 \cdot 9 = 45$, and $4 + 5 = 9$; $33 \cdot 9 = 297$, and $2 + 9 + 7 = 18$.) Therefore, $2 + a + b + 4 = 6 + a + b$ must equal either 9 or 18.

(2) If a number is divisible by 11, then if we alternately add and subtract its digits, we get a multiple of 11. (For example, $17 \cdot 11 = 187$, and $1 - 8 + 7 = 0$; $987 \cdot 11 = 10857$, and $1 - 0 + 8 - 5 + 7 = 11$.) Therefore $2 - a + b - 4$ must be a multiple of 11. Because a and b are between 0 and 9, either $a = 9$ and $b = 0$, or $b - a = 2$.

Because 290004 is not divisible by 9, we have two systems to solve: Either $a + b = 3$ and $b - a = 2$, or $a + b = 12$ and $b - a = 2$. The first gives $b = 2.5$ (which doesn't make sense), but the second leads to $b = 7$ and $a = 5$. The answer is therefore 250074.

Note that, if one knew either one of these facts, trial and error could be used to finish the problem. For example, once one has established that $a + b = 3$ or $a + b = 12$, one could try all such combinations of a and b (there are only 11).

9. Find the exact perimeter of quadrilateral $QUAD$ on the right.

Solution: $\triangle QUD$ is a 30–60–90 right triangle, so the hypotenuse has length $UD = 2\sqrt{2}$, and leg \overline{QU} has length $QU = \sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.

Meanwhile, $\triangle UAD$ is an isosceles (45–45–90) right triangle, so

$UA = UD = 2\sqrt{2}$ and $AD = 4$. Therefore, the perimeter is $QU + UA + AD + DQ = \sqrt{6} + 2\sqrt{2} + 4 + \sqrt{2} = \sqrt{6} + 3\sqrt{2} + 4$.

10. Solve for x : $\begin{vmatrix} 1 & x & 2 \\ 3 & 0 & 4 \\ -1 & 3 & -2 \end{vmatrix} = 2004$.

Solution: Expand the determinant of the matrix across the first row:

$$\begin{aligned} \begin{vmatrix} 1 & x & 2 \\ 3 & 0 & 4 \\ -1 & 3 & -2 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 0 & 4 \\ 3 & -2 \end{vmatrix} - x \cdot \begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} \\ &= 1 \cdot [(0)(-2) - (3)(4)] - x \cdot [(3)(-2) - (-1)(4)] + 2 \cdot [(3)(3) - (-1)(0)] \\ &= 1 \cdot (-12) - x \cdot (-2) + 2 \cdot (9) \\ &= 2x + 6 \end{aligned}$$

Therefore, $2x + 6 = 2004$, so $2x = 1998$, so $x = 999$.

11. Solve for x : $|2x + 5| - 11 = 2004$.

Solution: Add 11 to both sides: $|2x + 5| = 2015$. Then either $2x + 5 = -2015$ or $2x + 5 = 2015$, so $2x = -5 \pm 2015$, so $x = 1005$ or $x = -1010$.

12. Solve for all real values of x , correct to the nearest thousandth:

$$0.2x^3 + 0.5x^2 - x + 2003 = 2004.$$

Solution: Using a calculator, we can find that $x \doteq -3.517374\dots$, $x \doteq 1.804942\dots$, or $x \doteq -0.787568\dots$. To the nearest thousandth, these are -3.517 , 1.805 , and -0.788 .

Section C. Each problem is worth 3 points.

13. A set of five natural numbers has a mean, median, and mode. The mean is 2004. Two of the numbers are 1900 and 1963. If the median is 2 less than the mode, write the value of the mode.

Solution: Call the five numbers a, b, c, d, e , with $0 < a \leq b \leq c \leq d \leq e$ (so that the median is c). Because the mean is 2004, we know that $a + b + c + d + e = 10020$. Because the mode is two more than c , we must have $d = e = c + 2$, and $0 < a < b < c$. Therefore, $a + b + c + d + e = a + b + 3c + 4$, so $a + b + 3c = 10016$. Now $3c < a + b + 3c < 5c$, so $2003 < c < 3339$. This means that $a = 1900$ and $b = 1963$, so $c = 2051$ and the mode is 2053.

14. In right $\triangle ABC$, $m\angle A = 20.04^\circ$, and the right angle is at C . Find the numerical value of $\sin^2 A + \sin^2 B + \sin^2 C$.

Solution: Computing $(\sin 20.04^\circ)^2 + (\sin 69.96^\circ)^2 + (\sin 90^\circ)^2$ on your calculator should give 2. In fact, the size of $\angle A$ does not matter: $\sin 90^\circ = 1$, and $\angle A$ and $\angle B$ must be complementary, so $\sin B = \cos A$, and of course $\sin^2 A + \cos^2 A = 1$.

15. A square pyramid has a base with perimeter 48 cm, and slant height 10 cm. Find the volume of the pyramid. Be sure to include the correct units.

Solution: If the (square) base has a perimeter of 48 cm, then the side length is 12 cm. The slant height corresponds to the hypotenuse of a right triangle with one leg length 6 cm. The pyramid therefore has height 8 cm. The volume of a pyramid is given by the formula $V = \frac{1}{3}Ah$, where A is the area of the base. Therefore, $V = \frac{1}{3}(12^2)(8) = 384 \text{ cm}^3$.