BC CONTEST 2003 (*SOLUTIONS*)

PART I. 30 Minutes. NO CALCULATORS

Section A. Each correct answer is worth 1 point.

1. Find the sum of the first 6 prime numbers.

Solution: 2 + 3 + 5 + 7 + 11 + 13 = 41

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2. Give the official (and more common) name for a regular quadrilateral.

Solution: regular = equal sides and equal interior angles, so it's a square

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3. Express the repeating decimal $0.1\overline{7}$ as a ratio of two positive integers in simplest form.

Solution: let $x = 0.1\overline{7}$.

$$\begin{array}{r} 10x = 1.77777... \\ -(\ x = 0.1777...) \\ \hline 9x = 1.6 \end{array}$$

(There are variations on this beginning.)

$\frac{9x}{9} = \frac{1.6}{9}$ $\quad x = \frac{1.6}{9} = \frac{16}{90}$ $\quad$ The simplified answer is $\frac{8}{45}$

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4. If 2x + 1 = 2003, find the value of 3x – 1000.

Solution: Solve for x:

$$\begin{array}{r} 2x + 1 = 2003 \\ -1 \quad -1 \\ \hline 2x = 2002 \end{array}$$

$\frac{2x}{2} = \frac{2002}{2}$ $\quad x = 1001$ $\quad$ So $\quad 3x - 1000 = 2003$

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5. A non-isosceles triangle has integral sides of 4, 5, and x. Find all possible values of x.

Solution: The triangle inequality theorem says that the sum of any two sides must be greater than the third side. So $5 - 4 < x < 4 + 5$, or $1 < x < 9$.

But the sides are integral (they are integers), so x is in the set {2, 3, 4, 5, 6, 7, 8}.

But the triangle is non-isosceles (no two sides can be equal).

Therefore, x is in {2, 3, 6, 7, 8}.

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6. Using some or all of the digits 0 – 9 (no digit more than once), construct the largest possible six-digit odd number with a 9 in the tens place.

Solution:

____9_	9 must be in the tens place.
8___9_	Biggest possible number.
87659_	Biggest possible number (continued).
876593	Biggest possible number (continued), number must be odd.

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7. 7 + (– 7) = 0. This is an example of what basic property of addition?

Solution: Additive inverse property or Opposites property.

Section B. Each correct answer is worth 2 points.

8. If $(x^5 + x^4 + x - 5)$ is divided by $(x + 1)$, find the remainder.

Solution 1: (Long division):

$$\begin{array}{r} x^4 + 1 \\ x+1 \,\big)\, x^5 + x^4 + x - 5 \\ -(x^5 + x^4) \\ \hline x - 5 \\ -(x+1) \\ \hline -6 \end{array}$$

Remainder = – 6

Solution 2: (Synthetic division): Works when dividing by $x - c$. Rewrite $x + 1$ as $x - (-1)$.

$$\begin{array}{r|rrrrrr} -1 & 1 & 1 & 0 & 0 & 1 & -5 \\ & & -1 & 0 & 0 & 0 & -1 \\ \hline & 1 & 0 & 0 & 0 & 1 & -6 \end{array}$$

Solution 3: (Math knowledge):

The Remainder theorem says you can just plug in -1 to get the remainder:

Remainder = $(-1)^5 + (-1)^4 + (-1) - 5 = -6$

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9. Find the 2003rd digit after the decimal point in the decimal representation of 4/7.

Solution: Rational numbers (such as fractions) either stop or repeat. If you do long division far enough the pattern is clear:

$$0.57142857142857\ldots \quad \text{or} \quad 0.\overline{571428}$$

So it repeats every six numbers. Now do 2003 ÷ 6. There are several variations on what to d next. Here's one of them:

$$\begin{array}{r} 333 \\ 6\,\overline{)\,2003} \\ -1998 \\ \hline 5 \end{array}$$

With a remainder of 5, we choose the 5th repeated digit.

The answer is 2.

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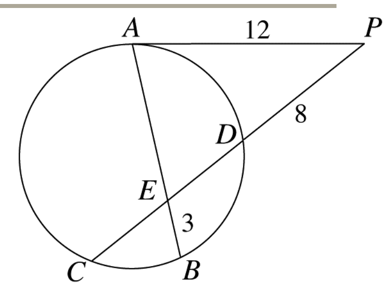
10. In the figure on the right, the length of tangent $\overline{AP}$ is 12, $PD = 8$, and chord $\overline{AB}$ bisects chord $\overline{CD}$. If $EB = 3$, find the length of $\overline{AE}$.



Solution: $AP*AP = PD*PC \;\rightarrow\; 12*12 = 8*PC \;\rightarrow\; PC = 18$

$AE*EB = DE*EC$ Since $\overline{AB}$ bisects chord $\overline{CD}$, $DE = EC$.

So $AE*EB = DE*DE$. Also $CD = PC - PD \;\rightarrow\; CD = 10 \;\rightarrow\; DE = 5$

Thus $AE*EB = DE*DE \;\rightarrow\; AE*3 = 5*5 \;\rightarrow\; AE = \dfrac{25}{3}$

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11. Express in simplest radical form (no radicals in the denominator): $\dfrac{3+\sqrt{2}}{4-2\sqrt{2}}$

Solution: This is sometimes called "rationalizing the denominator."

$$\frac{3+\sqrt{2}}{4-2\sqrt{2}} \times \frac{4+2\sqrt{2}}{4+2\sqrt{2}} = \frac{12+6\sqrt{2}+4\sqrt{2}+4}{16+8\sqrt{2}-8\sqrt{2}-8} = \frac{16+10\sqrt{2}}{8} = \frac{8+5\sqrt{2}}{4}$$

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12. Write the numerical value of x if $\log_8 128 = x$. Express in simplest form.

Solution: Transform to exponential form: $8^x = 128$

Rewrite both 8 and 128 as powers of 2: $\left(2^3\right)^x = 2^7 \;\rightarrow\; \left(2^3\right)^x = 2^7 \;\rightarrow\; 2^{3x} = 2^7$

$\rightarrow\; 3x = 7 \;\rightarrow\; x = \dfrac{7}{3}$

Section C. Each correct answer is worth 3 points.

13. The ellipse $x^2 + 2y^2 + 12y - 10x - 57 = 0$ has a major axis with two endpoints. Find the coordinates of the endpoint that lies in quadrant IV. Express in ordered pair form, (x,y).

Solution: Put in standard form using complete the square:

$$x^2 - 10x + 2y^2 + 12y = 57 \quad \rightarrow \quad x^2 - 10x + 5^2 + 2(y^2 + 6y + 3^2) = 57 + 5^2 + 2(3^2)$$

$$\rightarrow \quad (x-5)^2 + 2(y+3)^2 = 100 \quad \rightarrow \quad \frac{(x-5)^2}{100} + \frac{(y+3)^2}{50} = 1 \qquad \text{Center: } (5, -3)$$

$a^2 = 100 \rightarrow a = \pm 10$ The endpoint in quadrant IV is $(5+10,\ -3)$ or $(15,\ -3)$

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14. Softball player Berni Williams has 120 hits in 300 at-bats for a current batting average of .400. In today's game, she will have 5 at-bats. What is the probability that she will get *exactly* 2 hits?

Solution 1: (Binomial theorem): $\binom{5}{2}(0.4)^2(0.6)^3$ or ${}_5C_2\,(0.4)^2(0.6)^3$

$$\binom{5}{2}(0.4)^2(0.6)^3 = \frac{1080}{3125} = \frac{216}{625} = 0.3456$$

Solution 2: (Brute force): $(0.4)(0.4)(0.6)(0.6)(0.6) + (0.4)(0.6)(0.4)(0.6)(0.6)$

$+ (0.4)(0.6)(0.6)(0.4)(0.6) + (0.4)(0.6)(0.6)(0.6)(0.4) + (0.6)(0.4)(0.4)(0.6)(0.6)$

$+ (0.6)(0.4)(0.6)(0.4)(0.6) + (0.6)(0.4)(0.6)(0.6)(0.4) + (0.6)(0.6)(0.4)(0.4)(0.6)$

$+ (0.6)(0.6)(0.4)(0.6)(0.4) + (0.6)(0.6)(0.6)(0.4)(0.4)$

Solution 3: (Binomial theorem & polynomials): $(0.4 + 0.6)^5 =$

$$(0.4)^5 + 5(0.4)^4(0.6)^1 + 10(0.4)^3(0.6)^2 + 10(0.4)^2(0.6)^3 + 5(0.4)^1(0.6)^4 + (0.6)^5$$

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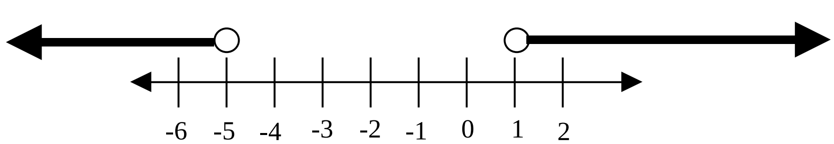
15. Solve the inequality $\left|\frac{\mathbf{x}+2}{3}\right| > 1$ and graph its solution on the given number line.

Solution: $\frac{x+2}{3} > 1$ or $\frac{x+2}{3} < -1$

Solving both for x gives: $x > 1$ or $x < -5$



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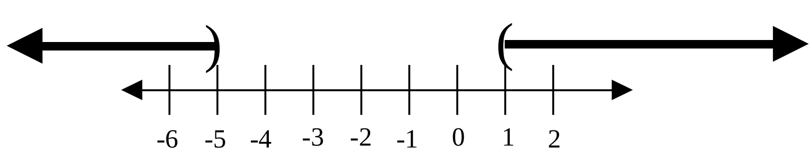

BC CONTEST 2003 **(*SOLUTIONS*)**

PART II. 30 Minutes. **CALCULATORS NEEDED**

Section A. Each problem is worth 1 point.

1. Find the exact average of the squares of the first six natural numbers.

Solution: $\frac{1+4+9+16+25+36}{6} = \frac{91}{6} = 15\frac{1}{6} = 15.1\overline{6}$

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2. On the first test you scored 80. On the second test you scored 89. Find the percent of increase to the nearest hundredth of one percent.

Solution: One way to think of a fraction is: $\frac{\textit{part you care about}}{\textit{original}}$

We have: $\frac{9}{80} = 0.1125$ Then change to a percent (multiply by 100): 11.25%

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3. Let a = $\log_2 8$; let b = $\log_3 8$. Find the value of a – b correct to the nearest hundredth.

Solution: $\log_{base}(X) = \log(X) / \log(\text{base})$

On full-display calculators, you can type: `log(8)/log(2) - log(8)/log(3)`

You get 1.107210739. To the nearest hundredth: 1.11

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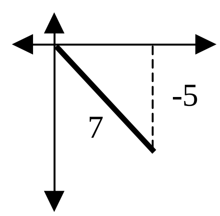
4. Let sin _ = $-\frac{5}{7}$, with terminal side in quadrant IV. Write the exact value of tan _.

Solution:

Use the pythagorean theorem:

$5^2 + x^2 = 7^2 \rightarrow x^2 = 24$

$x = \pm\sqrt{24} = \pm 2\sqrt{6}$ tan _ = $\frac{opp}{adj} = \frac{-5}{2\sqrt{6}}$ or $\frac{-5\sqrt{6}}{12}$



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5. Name the author of the first geometry book, *Elements*.

Solution: Euclid (Greek, around 300 B.C.).

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6. Otto's Auto Rental charges $52 per day, and 32¢ per mile for an automobile rental. Arnold paid $578.40 to rent a car for three days. How many miles did he drive?

Solution: The cost equation is: Cost = 52*days + 0.32*miles

Since he rented the car for three days, we can simplify the equation: $C = 156 + 0.32m$

Then put in his cost: $578.40 = 156 + 0.32m$

And solve for m: $422.40 = 0.32m \rightarrow 1320 = m$

7. Find the domain of the function: $\mathbf{f(x) = x^2 - 2}$

Solution 1: (Algebraic)

Only two things are bad when dealing with real numbers:

1) square root of a negative number

2) division by zero

Neither of these will happen, so the domain is all real numbers. Or $(-\infty, \infty)$.

Solution 2: (Graphing calculator)

The graph appears to extend in both the positive and negative *x*-directions forever. So we would guess that the domain is all real numbers.

Solution 3: (Math knowledge)

The domain of all polynomials is all real numbers.

Section B. Each problem is worth 2 points.

8. Write the equation of the line through (–3, 2) and is perpendicular to the line 3x – y = 7. Express your answer in slope-intercept form: $y = mx + b$.

Solution: There are many approaches to this problem. Here is one:

Put 3x – y = 7 in slope-intercept form: y = 3x + 7

Its slope is 3; a line perpendicular to it has a slope of $-\frac{1}{3}$.

So our equation has the form: $y = -\frac{1}{3}x + b$. Temporarily substitute the point (–3, 2):

$2 = -\frac{1}{3}(-3) + b$ → $b = 1$. So the equation is $y = -\frac{1}{3}x + 1$

9. The positive integers, *a, b, c* are said to form a Pythagorean Triple if $a^2 + b^2 = c^2$. Write <u>two</u> distinct Pythagorean Triples in the form *a,b,c* (with $a < b < c$) such that c = 65.

<u>**Solution 1**</u>: (A little algebra and trial & error)

Solve for b^2 and you get $b^2 = c^2 - a^2$ Trial and error: $\sqrt{65^2 - 1^2}$ not an integer, etc.

Eventually you get 16, 63, 65 and 25, 60, 65

Other answers include 33, 56, 65 and 39, 52, 65

(Quicken the trial and error by using a table or lists on the TI-82 & TI-83.)

<u>**Solution 2**</u>: (Math knowledge)

Notice that 65 = **5***13. That is a scale factor of 13 for a 3–4–5 triangle.

Thus one triple is: 39, 52, 65

Notice that 65 = **13***5. That is a scale factor of 5 for a 5–12–13 triangle.

Another triple is: 25, 60, 65

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10. $\mathbf{n!} = \sum_{\mathbf{k}=1}^{15} \mathbf{k}$. Write the value of n.

<u>**Solution**</u>: $\sum_{k=1}^{15} k = 120$ You can do this quickly in a calculator.

However, there is a formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ or $\frac{15(16)}{2}$ in this case.

A little trial and error reveals the answer: 120 = 5!

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11. Find three numbers *a*, *b*, and *c* so that 2, *a*, *b*, *c*, 5 is an arithmetic sequence.

<u>**Solution 1**</u>: The formula for the n^{th} term of an arithmetic sequence is $a_n = a_1 + d(n-1)$

If we use the formula for the 5th term, we know everything except *d* (the *difference*):

$5 = 2 + d(5-1) \quad \rightarrow \quad 5 = 2 + 4d \quad \rightarrow \quad 3 = 4d \quad \rightarrow \quad \frac{3}{4} = d$

$a = 2.75$ $b = 3.5$ $c = 4.25$.

<u>**Solution 2**</u>: The average of the 1st and 5th terms is *b*. $b = \frac{2+5}{2} = 3.5$

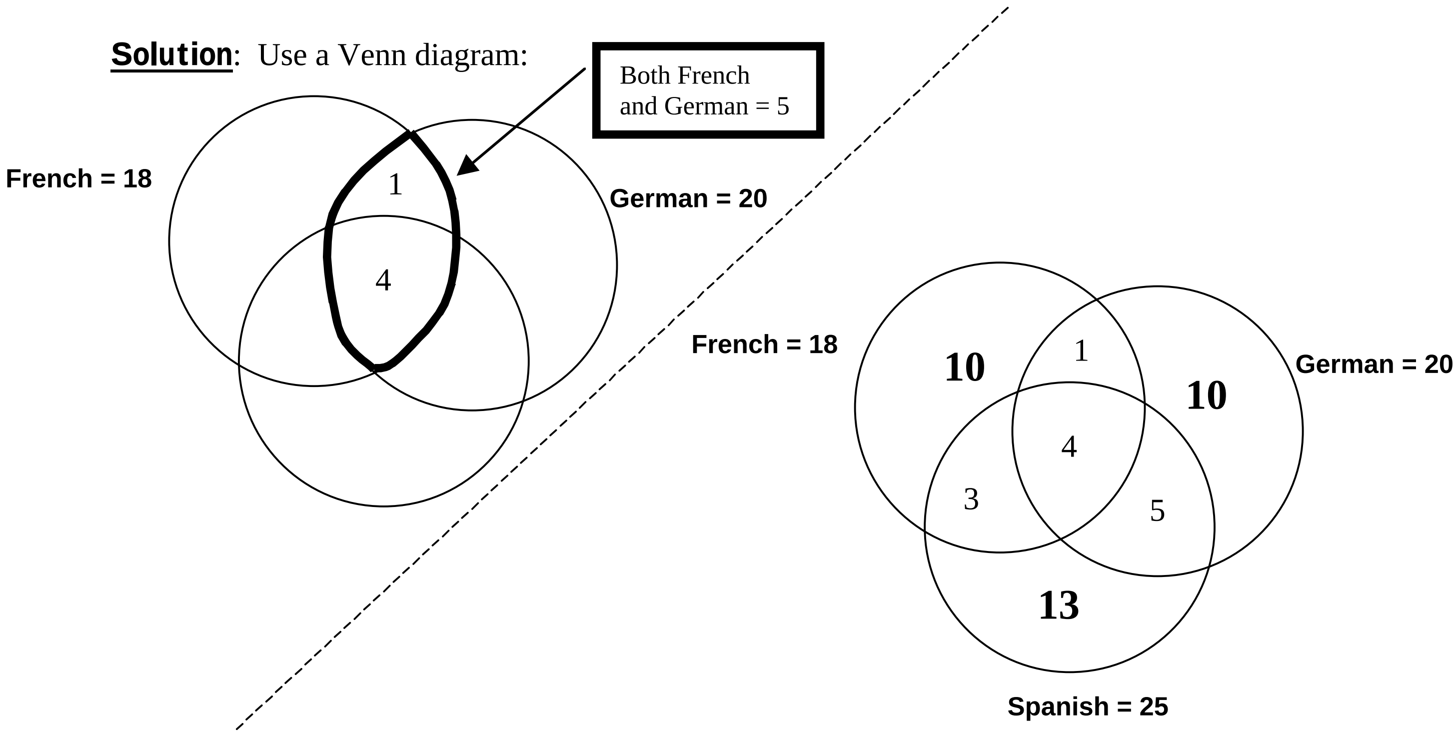
Likewise: $a = \frac{2+3.5}{2} = 2.75$ $c = \frac{3.5+5}{2} = 4.25$

12. In a certain class of 60 students:

20 study German; 9 study German and Spanish; 18 study French
5 study German and French; 25 study Spanish
7 study Spanish and French; 4 study all 3 languages

How many students are studying no language?

Solution: Use a Venn diagram:



That's a total of 10 + 10 + 13 + 1 + 5 + 3 + 4 = 46 students taking a language. Thus 14 are not taking a language.

Section C. Each correct answer is worth 3 points.

13. From the top of a 200-ft-tall square water tower, SpongeBob observes a car moving toward the tower. If the angle of depression of the car changes from 25° to 50° during the time of observation, how far does the car travel (to the nearest foot)?

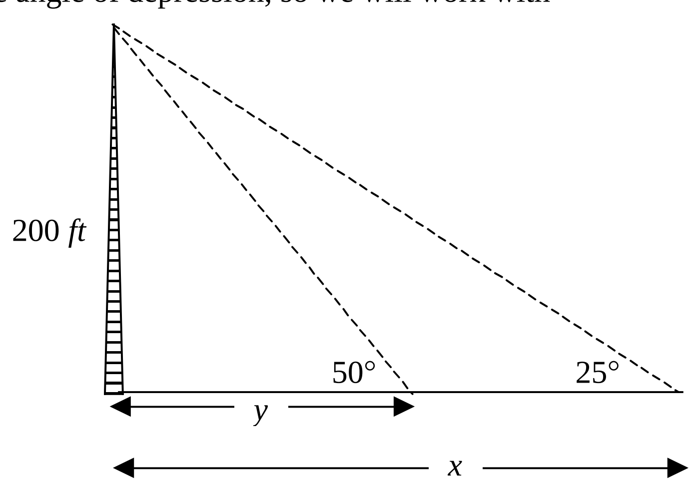
Solution: Angle of elevation always equals the angle of depression, so we will work with angle of elevation (it's easier).

$\tan(25^\circ) = \frac{200}{x} \quad \rightarrow \quad x = 428.9$

$\tan(50^\circ) = \frac{200}{y} \quad \rightarrow \quad y = 167.8$

Travel distance = $x - y$ = 261.1

Car travels 261 feet.



14. Given matrix A = $\begin{matrix} 3 & 3 & -1 \\ 2 & \mathbf{x} & 5 \end{matrix}$ and matrix B = $\begin{matrix} 9 & 2 \\ 4 & 3 \\ -7 & 1 \end{matrix}$. The value of the entry in the second row, first column of matrix AB is 2003. Find x.

Solution: The second row, first column comes from multiplying this row and column:

$$\begin{matrix} 3 & 3 & -1 \\ \boxed{2 \quad \mathbf{x} \quad 5} \end{matrix} \qquad \begin{matrix} 9 & 2 \\ 4 & 3 \\ -7 & 1 \end{matrix} \quad \rightarrow \quad 2(9) + x(4) + 5(-7) \quad \rightarrow \quad 4x - 17$$

Set this equal to the given entry value and solve for x: $4x - 17 = 2003 \quad \rightarrow \quad x = 505$

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15. A right circular cone has the same volume as a sphere. If the radius of the base of the cone has the same radius as the sphere, find the height of the cone in terms of r.

Solution: Volume of **any** cone: $V = \frac{1}{3}\pi r^2 h$ Volume of sphere: $V = \frac{4}{3}\pi r^3$

Since they have the same volume, set the equations equal to each other (or use substitution):

$$\frac{1}{3}\pi r^2 h = \frac{4}{3}\pi r^3$$

> *Note*: If the radii were not the same, we would need to use two different variables for r (perhaps r_1 and r_2)

Now work on getting h by itself. There are many approaches! Here is one approach:

Multiply both sides by 3: $\pi r^2 h = 4\pi r^3$ Divide both sides by π: $r^2 h = 4r^3$

Divide both sides by r^2: $h = 4r$